SIGNED YOUNG'S RULE AND SEMISTANDARD HOMOMORPHISMS

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Abstract. We show that a signed Young permutation module $M(\alpha|\beta)$ has a Specht filtration in which a Specht module S^λ appears with a multiplicity given by the number of semistandard λ-tableaux of type $(\alpha|\beta)$. We also construct a class of homomorphisms from a Specht module to a signed Young permutation module which generalise James' semistandard homomorphisms. The semistandard homomorphisms we have constructed form a basis for the space $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, M(\alpha|\beta))$ in the semisimple case.

1. Introduction

Modular representation theory of finite groups, unlike its ordinary counterpart, is not at all well understood. Even for ubiquitous groups like the symmetric groups, many fundamental questions remain open. It is therefore important to understand the naturally occurring representations, as these may provide more information about the modular representation theory in general.

Among the naturally occurring representations of the symmetric groups, the Young permutation modules are perhaps the most well-known. James [J] studied these modules in detail and exploited their knowledge to obtain important information such as those about the Specht modules.

When the characteristic is not 2, the signed Young permutation modules are a natural generalisation of the Young permutation modules. Their indecomposable summands, known as signed Young modules, are first studied by Donkin in [D]. Subsequently, signed Young modules are shown to be related to irreducible Specht modules. More specifically, Hemmer [H] showed that irreducible Specht modules are signed Young modules and, recently, Danz and the first author [DL] described the label explicitly.

In this paper, we first show that there is a signed Young's rule for the signed Young permutation modules, i.e. the signed Young permutation module $M(\alpha|\beta)$ has a Specht filtration in which the multiplicity of S^λ as a factor equals the number of semistandard λ-tableaux of type $(\alpha|\beta)$. The idea of our proof is fairly straightforward, using facts which are well-known among experts.

Building on this success, we then attempt to construct homomorphisms from the Specht module S^λ to the signed Young permutation module $M(\alpha|\beta)$, just like James did for the (unsigned) Young permutation modules. Unfortunately, this is

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no longer easy or straightforward. Coming up with a conjectural generalisation of James's construction of his homomorphisms took more than a month of intensive work, even before we could begin to prove that it is indeed correct. We faced further obstruction when we tried to show that a subset of these homomorphisms, corresponding to the semistandard λ-tableaux of type $(\alpha|\beta)$, gives a basis for the homomorphism space. We found many examples in our study which show that the signed Young permutation modules behave in a much more unpredictable way than the unsigned ones. We include some of these in Examples 4.1, 5.2 and 5.3.

Our signed Young's rule for the signed Young permutation modules has already an application: Danz and the first named author [DL] use this fact in their determination of the exact label of a signed Young module which is isomorphic to an irreducible Specht module.

The paper is organised as follows. In the next section, we give a quick introduction of the background. In Section 3, we prove the signed Young's rule for signed Young permutation modules. In Section 4, we generalise James's construction to obtain some homomorphisms from the Specht module S^λ to the signed Young permutation module $M(\alpha|\beta)$, and in Section 5, we look into the conditions for which a subset of our constructed homomorphisms, corresponding to the semistandard λ-tableaux of type $(\alpha|\beta)$, will be a basis for $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda, M(\alpha|\beta))$.

2. Preliminaries

In this section, we provide the necessary background and introduce some notations that we shall use in this paper.

Throughout, we fix a field $\mathbb{F}$ of arbitrary characteristic.

2.1. **Symmetric groups.** Let X be a finite set. The symmetric group $\mathfrak{S}_X$ on X is the group of bijections from X to X under composition of functions. By convention, $\mathfrak{S}_\emptyset$ is the trivial group. When Y is a non-empty subset of X, we view $\mathfrak{S}_Y$ as a subgroup of $\mathfrak{S}_X$ by identifying an element of $\mathfrak{S}_Y$ with its extension that sends x to x for all $x \in X \setminus Y$.

Let $X \subseteq \mathbb{Z}^+$, and $k \in \mathbb{Z}^+$. Define the k-translated subset X^{+k} of $\mathbb{Z}^+$ by

$$X^{+k} = \{x + k : x \in X\}.$$

Let $\mathfrak{S}_X^{+k} = \{f^{+k} : f \in \mathfrak{S}_X\}$ where $f^{+k} : X^{+k} \to X^{+k}$ is defined by

$$f^{+k}(x + k) = f(x) + k$$

for all $x \in X$. Clearly, $f \mapsto f^{+k}$ is a group isomorphism from $\mathfrak{S}_X$ to $\mathfrak{S}_X^{+k}$. For any $S \subseteq \mathfrak{S}_X$, write S^{+k} for $\{s^{+k} : s \in S\}$.

For $n \in \mathbb{Z}^+$, we write $\mathfrak{S}_n$ for $\mathfrak{S}_{\{1,2,\dots,n\}}$, the usual symmetric group on n letters.

2.2. **Compositions and Young subgroups.** A composition is a finite sequence of positive integers. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a composition. Then $\ell(\lambda) = r$. If $\sum_{i=1}^{\ell(\lambda)} \lambda_i = n$, we say that λ is a composition of n, and write $|\lambda| = n$. If in addition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$, then we call λ a partition of n. The unique composition (or partition) of 0 is denoted by $\varnothing$, and $\ell(\varnothing) = 0$.

To a composition λ of n, we associate the Young subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$, where

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1}\mathfrak{S}_{\lambda_2}^{+s(\lambda)_1}\cdots\mathfrak{S}_{\lambda_{\ell(\lambda)}}^{+s(\lambda)_{\ell(\lambda)-1}} \cong \mathfrak{S}_{\lambda_1}\times\mathfrak{S}_{\lambda_2}\times\cdots\times\mathfrak{S}_{\lambda_{\ell(\lambda)}}$$

and $s(\lambda)_j = \sum_{i=1}^{j}\lambda_i$ for all $1 \le j \le \ell(\lambda)-1$.

Let $(\alpha|\beta)$ be a bicomposition of n, i.e. α and β are compositions and $|\alpha|+|\beta| = n$. One may view $(\alpha|\beta)$ as a composition of n by concatenating α with β, and so we have the associated Young subgroup

$$\mathfrak{S}_{\alpha|\beta} = \mathfrak{S}_\alpha\mathfrak{S}_\beta^{+|\alpha|} \cong \mathfrak{S}_\alpha\times\mathfrak{S}_\beta.$$

2.3. **Young diagrams.** For a partition λ, its Young diagram $[\lambda]$ is defined as

$$[\lambda] = \{(i,j)\in\mathbb{Z}^2 : 1\le i\le \ell(\lambda),\ 1\le j\le\lambda_i\}.$$

The conjugate partition of λ, denoted λ', is the partition whose Young diagram $[\lambda'] = \{(i,j) : (j,i)\in[\lambda]\}$.

If ξ is a subpartition of λ (i.e. ξ is a partition and $[\xi]\subseteq[\lambda]$), then the skew diagram $[\lambda/\xi]$ is defined to be $[\lambda]\setminus[\xi]$.

2.4. **Tableaux.** Let λ be a partition of n. Define formally a λ-tableau $\mathfrak{t}$ to be a bijective function $\mathfrak{t} : [\lambda]\to\{1,2,\dots,n\}$. We usually view this as a labelling of the elements of $[\lambda]$ by numbers $1,2,\dots,n$, such that each number appears exactly once. Denote the set of λ-tableaux by $\mathscr{T}(\lambda)$.

Through a fixed λ-tableau $\mathfrak{t}$ we obtain a group isomorphism $\mathfrak{S}_n\cong\mathfrak{S}_{[\lambda]}$ where we identify $\sigma\in\mathfrak{S}_n$ with $\mathfrak{t}^{-1}\circ\sigma\circ\mathfrak{t}\in\mathfrak{S}_{[\lambda]}$. Let $R_{[\lambda]}$ and $C_{[\lambda]}$ be the row and column stabilizers of $[\lambda]$, i.e.

$$\begin{aligned} R_{[\lambda]} &= \{f\in\mathfrak{S}_{[\lambda]} : \forall(i,j)\in[\lambda], \exists(i,j')\in[\lambda], f(i,j) = (i,j')\},\\ C_{[\lambda]} &= \{f\in\mathfrak{S}_{[\lambda]} : \forall(i,j)\in[\lambda], \exists(i',j)\in[\lambda]\ f(i,j) = (i',j)\}.\end{aligned}$$

Under the above group isomorphism, $R_{[\lambda]}$ corresponds to the row stabilizer $R_\mathfrak{t}$ of $\mathfrak{t}$, i.e. $R_\mathfrak{t} = \mathfrak{t}\circ R_{[\lambda]}\circ\mathfrak{t}^{-1}\subseteq\mathfrak{S}_n$, and similarly, $C_{[\lambda]}$ corresponds to the column stabilizer $C_\mathfrak{t}$ of $\mathfrak{t}$, i.e. $C_\mathfrak{t} = \mathfrak{t}\circ C_{[\lambda]}\circ\mathfrak{t}^{-1}$.

Let $\mathfrak{t}^\lambda$ denote the *initial λ-tableau*, defined by $\mathfrak{t}^\lambda(i,j) = \lambda_1+\cdots+\lambda_{i-1}+j$ for all $(i,j)\in[\lambda]$. Observe that $R_{\mathfrak{t}^\lambda} = \mathfrak{S}_\lambda$.

Post-composition of λ-tableaux by elements of the symmetric group $\mathfrak{S}_n$ gives a well-defined, faithful and transitive left action of $\mathfrak{S}_n$ on $\mathscr{T}(\lambda)$, i.e. $\sigma\cdot\mathfrak{t} = \sigma\circ\mathfrak{t}$ for all $\sigma\in\mathfrak{S}_n$ and $\mathfrak{t}\in\mathscr{T}(\lambda)$. Observe that

$$R_{\sigma\cdot\mathfrak{t}} = R_{\sigma\circ\mathfrak{t}} = (\sigma\circ\mathfrak{t})\circ R_{[\lambda]}\circ(\sigma\circ\mathfrak{t})^{-1} = \sigma\circ R_\mathfrak{t}\circ\sigma^{-1} = \sigma R_\mathfrak{t}\sigma^{-1},$$

and similarly, $C_{\sigma\cdot\mathfrak{t}} = \sigma C_\mathfrak{t}\sigma^{-1}$. In particular, the row stabilizers of the λ-tableaux are conjugate subgroups of $\mathfrak{S}_\lambda$.

A λ-tableau $\mathfrak{t}$ is *standard* if it is increasing along each row and down each column i.e. we have both $\mathfrak{t}(i,j)<\mathfrak{t}(i,j')$ and $\mathfrak{t}(i,j)<\mathfrak{t}(i',j)$ for all $(i,j),(i,j'),(i',j)\in[\lambda]$ with $j<j'$ and $i<i'$. Write $\mathscr{T}_{\mathrm{std}}(\lambda)$ for the set of all standard λ-tableaux.

Let $(\alpha|\beta)$ be a bicomposition of n and λ be a partition of n. We view a λ-tableau T of type $(\alpha|\beta)$ as a colouring of the nodes of $[\lambda]$ by the colours $\mathbb{c}_1,\mathbb{c}_2,\dots,\mathbb{c}_{\ell(\alpha)},\mathbb{d}_1,\mathbb{d}_2,\dots,\mathbb{d}_{\ell(\beta)}$

such that there are exactly α_i nodes of colour $\mathbb{c}_i$ and β_j nodes of colour $\mathbb{d}_j$ for all $1 \le i \le \ell(\alpha)$ and $1 \le j \le \ell(\beta)$. Formally, T is a function

$$\mathrm{T} : [\lambda] \to \{\mathbb{c}_1, \mathbb{c}_2, \dots, \mathbb{c}_{\ell(\alpha)}, \mathbb{d}_1, \mathbb{d}_2, \dots, \mathbb{d}_{\ell(\beta)}\}$$

such that $|\mathrm{T}^{-1}(\{\mathbb{c}_i\})| = \alpha_i$ and $|\mathrm{T}^{-1}(\{\mathbb{d}_j\})| = \beta_j$ for all $1 \le i \le \ell(\alpha)$ and $1 \le j \le \ell(\beta)$. We write $\mathscr{T}(\lambda, (\alpha|\beta))$ for the set of all λ-tableaux of type $(\alpha|\beta)$.

Let $\mathfrak{t}_0 \in \mathscr{T}(\lambda)$ be a fixed λ-tableau (which may or may not be the initial λ-tableau $\mathfrak{t}^\lambda$ defined above). We define the canonical λ-tableau of type $(\alpha|\beta)$ associated to $\mathfrak{t}_0$, denoted as T_0, as follows:

$$\mathrm{T}_0(i,j) = \begin{cases} \mathbb{c}_k & \text{if } \sum_{a=1}^{k-1} \alpha_a < \mathfrak{t}_0(i,j) \le \sum_{a=1}^{k} \alpha_a \text{ for some } k, \\ \mathbb{d}_k & \text{if } |\alpha| + \sum_{b=1}^{k-1} \beta_b < \mathfrak{t}_0(i,j) \le |\alpha| + \sum_{b=1}^{k} \beta_b \text{ for some } k; \end{cases}$$

namely, the nodes labelled $1, \dots, \alpha_1$ by $\mathfrak{t}_0$ are coloured $\mathbb{c}_1$, those labelled $\alpha_1 + 1, \dots, \alpha_1 + \alpha_2$ are coloured $\mathbb{c}_2$, and so on.

We also have a transitive left action on $\mathscr{T}(\lambda, (\alpha|\beta))$ by $\mathfrak{S}_n$ through $\mathfrak{t}_0$ as follows: $\sigma \cdot \mathrm{T} = \mathrm{T} \circ (\mathfrak{t}_0)^{-1} \circ \sigma^{-1} \circ \mathfrak{t}_0$ for all $\sigma \in \mathfrak{S}_n$ and $\mathrm{T} \in \mathscr{T}(\lambda, (\alpha|\beta))$. Thus, if $(i,j) \in [\lambda]$ is labelled a by $\mathfrak{t}_0$, while the node labelled $\sigma^{-1}(a)$ by $\mathfrak{t}_0$ is coloured $\mathbb{c}_k$ (respectively, $\mathbb{d}_k$) by T, then $\sigma \cdot \mathrm{T}$ colours (i,j) with $\mathbb{c}_k$ (respectively, $\mathbb{d}_k$). Observe that under this action, the stabiliser of T_0 is $\mathfrak{S}_{\alpha|\beta}$. Consequently, $\mathscr{T}(\lambda, (\alpha|\beta))$ is in a bijective correspondence with the set of left cosets of $\mathfrak{S}_{\alpha|\beta}$ in $\mathfrak{S}_n$ via $d \cdot \mathrm{T}_0 \leftrightarrow d\mathfrak{S}_{\alpha|\beta}$. We set

$$\mathrm{T}_d = d \cdot \mathrm{T}_0$$

so that $\mathrm{T}_d = \mathrm{T}_{d'}$ if and only if $d^{-1}d' \in \mathfrak{S}_{\alpha|\beta}$.

We order the colours $\mathbb{c}_1, \mathbb{c}_2, \dots, \mathbb{d}_1, \mathbb{d}_2, \dots$ as follows:

$$\mathbb{c}_1 < \mathbb{c}_2 < \cdots < \mathbb{d}_1 < \mathbb{d}_2 < \cdots .$$

A λ-tableau T of type $(\alpha|\beta)$ is *row semistandard* (respectively, *column semistandard*) if every row (respectively, column) of T is weakly increasing. More precisely, T is row semistandard if $\mathrm{T}(i,j) \le \mathrm{T}(i,j')$ for all $(i,j), (i,j') \in [\lambda]$ with $j < j'$, and T is column semistandard if $\mathrm{T}(i,j) \le \mathrm{T}(i',j)$ for all $(i,j), (i',j) \in [\lambda]$ with $i < i'$.

Definition 2.1. Let λ be a composition of n and $(\alpha|\beta)$ be a bicomposition of n. A λ-tableau T of type $(\alpha|\beta)$ is *semistandard* if and only if

(i) T is both row and column semistandard;
(ii) $\mathrm{T}(i,j) = \mathrm{T}(i,j')$ for some $(i,j), (i,j') \in [\lambda]$ with $j \ne j'$ only if $\mathrm{T}(i,j) = \mathbb{c}_k$ for some k;
(iii) $\mathrm{T}(i,j) = \mathrm{T}(i',j)$ for some $(i,j), (i',j) \in [\lambda]$ with $i \ne i'$ only if $\mathrm{T}(i,j) = \mathbb{d}_k$ for some k.

The set of semistandard λ-tableaux of type $(\alpha|\beta)$ is denoted by $\mathscr{T}_{\mathrm{sstd}}(\lambda, (\alpha|\beta))$.

It is easy to see that the above discussion generalises the notion of λ-tableaux of type μ in the classical case, in the sense that a (semistandard) λ-tableau of type μ is a (semistandard) λ-tableau of type $(\mu|\varnothing)$. We denote by $\mathscr{T}(\lambda, \mu) =$

$\mathscr{T}(\lambda,(\mu|\varnothing))$ the set of λ-tableaux of type μ and by $\mathscr{T}_{\mathrm{sstd}}(\lambda,\mu)=\mathscr{T}_{\mathrm{sstd}}(\lambda,(\mu|\varnothing))$ the set of semistandard λ-tableaux of type μ. By convention, $|\mathscr{T}(\varnothing,(\varnothing|\varnothing))|=1$.

Our definition of (semistandard) λ-tableaux of type μ (and more generally of type $(\alpha|\beta)$) generalises in an obvious manner to that of (semistandard) λ/ξ-tableaux of type γ when ξ is a subpartition of a partition λ and $|\gamma|=|\lambda|-|\xi|$. As such, we also have both sets $\mathscr{T}(\lambda/\xi,\gamma)$ and $\mathscr{T}_{\mathrm{sstd}}(\lambda/\xi,\gamma)$.

2.5. **One-dimensional representations of symmetric groups.** For the remainder of this section, let $\mathcal{O}$ be either $\mathbb{F}$ or $\mathbb{Z}$. The signature representation of the symmetric group $\mathfrak{S}_n$ over $\mathcal{O}$ is denoted by $\mathrm{sgn}:\mathfrak{S}_n\to\{\pm1\}\subseteq\mathcal{O}$. We shall abuse notation and also write sgn for the $\mathcal{O}\mathfrak{S}_n$-module associated to it. In this latter context, sgn is $\mathcal{O}$-free of rank 1, with basis $\{\epsilon\}$. We denote the trivial $\mathcal{O}\mathfrak{S}_n$-module that is $\mathcal{O}$-free of rank 1 as $\mathcal{O}$, with basis $\{\mathbb{1}\}$. Thus, $\sigma\cdot\mathbb{1}=\mathbb{1}$ and $\sigma\cdot\epsilon=\mathrm{sgn}(\sigma)\epsilon$ for all $\sigma\in\mathfrak{S}_n$.

When H is a subgroup of $\mathfrak{S}_n$, we write the respective restricted $\mathcal{O}H$-modules as $\mathcal{O}_H$ and sgn_H. Sometimes, we shall abuse notation and write them as $\mathcal{O}$ and sgn when there is no confusion.

2.6. **Young permutation modules and Specht modules.** Let μ be a composition of n. The Young permutation module $M^\mu=M^\mu_{\mathcal{O}}$ is the permutation module associated to the left regular action of $\mathfrak{S}_n$ on the left cosets of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$. In other words, $M^\mu=\bigoplus_{d\mathfrak{S}_\mu\in\mathfrak{S}_n/\mathfrak{S}_\mu}\mathcal{O}(d\mathfrak{S}_\mu)$, and $\sigma\cdot(d\mathfrak{S}_\mu)=(\sigma d)\mathfrak{S}_\mu$ for all $\sigma\in\mathfrak{S}_n$ and $d\mathfrak{S}_\mu\in\mathfrak{S}_n/\mathfrak{S}_\mu$.

When $\tilde{\mu}$ is a composition of n obtained by rearranging some parts of μ, the Young subgroups $\mathfrak{S}_\mu$ and $\mathfrak{S}_{\tilde{\mu}}$ are conjugate in $\mathfrak{S}_n$, so that $M^\mu\cong M^{\tilde{\mu}}$ as $\mathcal{O}\mathfrak{S}_n$-modules.

Let λ be a partition. Given a λ-tableau $\mathfrak{t}$, let $d_{\mathfrak{t}}=\mathfrak{t}\circ(\mathfrak{t}^\lambda)^{-1}$. Then $d_{\mathfrak{t}}\in\mathfrak{S}_n$ and $d_{\mathfrak{t}}\cdot\mathfrak{t}^\lambda=d_{\mathfrak{t}}\circ\mathfrak{t}^\lambda=\mathfrak{t}$. We define the polytabloid

$$e_{\mathfrak{t}}=\sum_{\sigma\in C_{\mathfrak{t}}}\mathrm{sgn}(\sigma)(\sigma\cdot(d_{\mathfrak{t}}\mathfrak{S}_\lambda))\in M^\lambda.$$

The Specht module $S^\lambda=S^\lambda_{\mathcal{O}}$ is defined to be the $\mathcal{O}$-submodule of M^λ spanned by $\{e_{\mathfrak{t}}:\mathfrak{t}\in\mathscr{T}(\lambda)\}$. It is not difficult to see that $\tau\cdot e_{\mathfrak{t}}=e_{\tau\cdot\mathfrak{t}}$ for $\tau\in\mathfrak{S}_n$ and $\mathfrak{t}\in\mathscr{T}(\lambda)$, so that S^λ is an $\mathcal{O}\mathfrak{S}_n$-submodule of M^λ.

Let $\mathcal{O}=\mathbb{F}$ and M^* denote the contragradient dual of an $\mathbb{F}\mathfrak{S}_n$-module M. The following isomorphism is well known (see [J, 8.15 Theorem]):

$$S^\lambda\otimes\mathrm{sgn}\cong(S^{\lambda'})^*.$$

When $\mathcal{O}$ is a field of characteristic 0, or p with $p>n$, the Specht modules S^λ, as λ runs over all the partitions of n, give a complete list of pairwise non-isomorphic irreducible $\mathcal{O}\mathfrak{S}_n$-modules.

2.7. **Signed Young permutation modules.** Let $(\alpha|\beta)$ be a bicomposition of n. We define the signed Young permutation module $M(\alpha|\beta)=M_{\mathcal{O}}(\alpha|\beta)$ to be

$$M(\alpha|\beta)=\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\alpha\times\mathfrak{S}_\beta}(\mathcal{O}\boxtimes\mathrm{sgn}),$$

where $\mathfrak{S}_\alpha \times \mathfrak{S}_\beta$ is identified with the Young subgroup $\mathfrak{S}_{\alpha|\beta}$ of $\mathfrak{S}_n$. Thus, when Γ is a left transversal of $\mathfrak{S}_{\alpha|\beta}$ in $\mathfrak{S}_n$, $M(\alpha|\beta)$ has a basis $\{d \otimes \mathbb{1} \otimes \epsilon : d \in \Gamma\}$, and $\sigma \cdot (d \otimes \mathbb{1} \otimes \epsilon) = \mathrm{sgn}(\xi_\beta)(d' \otimes \mathbb{1} \otimes \epsilon)$ if $\sigma d = d' \xi_\alpha \xi_\beta^{+|\alpha|}$ where $d' \in \Gamma$, $(\xi_\alpha, \xi_\beta) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$.

Observe that if $\tilde{\alpha}$ and $\tilde{\beta}$ are compositions of $|\alpha|$ and $|\beta|$ obtained by rearranging some parts of α and β respectively, then $M(\tilde{\alpha}|\tilde{\beta}) \cong M(\alpha|\beta)$. Also,

$$M(\alpha|\beta) \otimes \mathrm{sgn} \cong \mathrm{Ind}_{\mathfrak{S}_\alpha \times \mathfrak{S}_\beta}^{\mathfrak{S}_n}((\mathcal{O} \otimes \mathrm{sgn}) \boxtimes (\mathrm{sgn} \otimes \mathrm{sgn})) \cong M(\beta|\alpha).$$

The Young permutation module M^μ, where μ is a composition of n, in the previous subsection is isomorphic to $\mathrm{Ind}_{\mathfrak{S}_\mu}^{\mathfrak{S}_n} \mathcal{O}$. Thus, $M^\mu \cong M(\mu|\varnothing)$. As such, signed Young permutation modules generalise Young permutation modules.

2.8. **Specht filtrations.** A filtration of an $\mathcal{O}\mathfrak{S}_n$-module M is a Specht filtration if every successive quotient is isomorphic to some Specht module. For our discussion in the next section, we need the following results of James and James-Peel in arbitrary characteristic.

Theorem 2.2 ([J, 17.14 Corollary]). *Let ξ be a partition of m, and let μ be a composition, with $m + |\mu| = n$. Then*

$$\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{|\mu|}}^{\mathfrak{S}_n}(S_\mathbb{F}^\xi \boxtimes M_\mathbb{F}^\mu)$$

has a Specht filtration in which all the factors are of the form $S_\mathbb{F}^\lambda$ for partitions λ of n such that $[\xi] \subseteq [\lambda]$, and the multiplicity of such a $S_\mathbb{F}^\lambda$ occurring as a factor of this filtration equals $|\mathscr{T}_{\mathrm{sstd}}(\lambda/\xi, \mu)|$.

Corollary 2.3. *Let ξ be a partition of m, and let μ be a composition, with $m + |\mu| = n$. Let $\tilde{\mu}$ be a composition obtained by rearranging some parts of μ. For any partition λ of n such that $[\xi] \subseteq [\lambda]$, we have*

$$|\mathscr{T}_{\mathrm{sstd}}(\lambda/\xi, \tilde{\mu})| = |\mathscr{T}_{\mathrm{sstd}}(\lambda/\xi, \mu)|.$$

Proof. When $\mathbb{F} = \mathbb{Q}$, the Specht modules are the irreducible modules, so that the Specht filtration in Theorem 2.2 is in fact a composition series. As such, $|\mathscr{T}_{\mathrm{sstd}}(\lambda/\xi, \mu)|$ (respectively, $|\mathscr{T}_{\mathrm{sstd}}(\lambda/\xi, \tilde{\mu})|$) counts the composition multiplicity of $S_\mathbb{Q}^\lambda$ in $\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{|\mu|}}^{\mathfrak{S}_n}(S_\mathbb{Q}^\xi \boxtimes M_\mathbb{Q}^\mu)$ (respectively, $\mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{|\mu|}}^{\mathfrak{S}_n}(S_\mathbb{Q}^\xi \boxtimes M_\mathbb{Q}^{\tilde{\mu}})$). Since $M_\mathbb{Q}^\mu \cong M_\mathbb{Q}^{\tilde{\mu}}$ as $\mathbb{Q}\mathfrak{S}_{|\mu|}$-modules, the corollary follows. □

For a partition λ of n and a bipartition $(\mu|\nu)$ of n, recall that the Littlewood-Richardson coefficient $c_{\mu\nu}^\lambda$ (by convention, $c_{\varnothing,\varnothing}^\varnothing = 1$) counts certain λ/μ-tableaux of type ν, and the classical Littlewood-Richardson's rule asserts that when $\mathbb{F}$ has characteristic 0, the multiplicity of the Specht module $S_\mathbb{F}^\lambda$ as a composition factor of $\mathrm{Ind}_{\mathfrak{S}_{|\mu|} \times \mathfrak{S}_{|\nu|}}^{\mathfrak{S}_n}(S_\mathbb{F}^\mu \boxtimes S_\mathbb{F}^\nu)$ (where $\mathfrak{S}_{|\mu|} \times \mathfrak{S}_{|\nu|}$ is identified with the subgroup $\mathfrak{S}_{|\mu|}\mathfrak{S}_{|\nu|}^{+|\mu|}$ of $\mathfrak{S}_n$) is precisely $c_{\mu\nu}^\lambda$. James and Peel generalise this to arbitrary fields, as follows:

Theorem 2.4 (James-Peel [JP, 5.5 Theorem]). *Let $(\mu|\nu)$ be a bipartition of n. Then $\mathrm{Ind}_{\mathfrak{S}_{|\mu|} \times \mathfrak{S}_{|\nu|}}^{\mathfrak{S}_n}(S_\mathbb{F}^\mu \boxtimes S_\mathbb{F}^\nu)$ has a Specht filtration. For a partition λ of n, the multiplicity of $S_\mathbb{F}^\lambda$ occurring as a factor of this filtration equals $c_{\mu\nu}^\lambda$.*

3. Signed Young's rule

The classical Young's rule states that when the field $\mathbb{F}$ has characteristic zero, the multiplicity of the (irreducible) Specht module $S^\lambda_\mathbb{F}$ occurring as a composition factor of the Young permutation module $M^\mu_\mathbb{F}$, where λ and μ are partitions of n, equals $|\mathscr{T}_{\mathrm{sstd}}(\lambda,\mu)|$, the number of semistandard λ-tableaux of type μ. In this section, we prove a signed Young's rule for the signed Young permutation module. More precisely, for arbitrary field $\mathbb{F}$, we show that the signed Young permutation module $M_\mathbb{F}(\alpha|\beta)$, where $(\alpha|\beta)$ is a bicomposition of n, has a Specht filtration in which the multiplicity of $S^\lambda_\mathbb{F}$ occurring as a factor of the filtration equals $|\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))|$, the number of semistandard λ-tableaux of type $(\alpha|\beta)$.

First, we need some general results, the first of which is well-known:

Lemma 3.1. *Let G be a finite group and let $0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$ be a filtration of an $\mathbb{F}G$-module M. Then we obtain a filtration*

$$0 = M_0^\perp \subseteq M_1^\perp \subseteq \cdots \subseteq M_r^\perp = M^*$$

of the dual $\mathbb{F}G$-module M^*, where $M_i^\perp = \{f \in M^* : f(M_i) = 0\}$ for each $i = 0,\ldots,r$. Furthermore, $M_{i+1}^\perp/M_i^\perp \cong (M_i/M_{i+1})^*$.

Lemma 3.2. *Let G and H be finite groups, and let M and N be $\mathbb{F}G$- and $\mathbb{F}H$-modules respectively. If M and N have filtrations, of lengths r and s, and with factors $E_1,\ldots,E_r$ and $F_1,\ldots,F_s$ respectively, then the $\mathbb{F}(G\times H)$-module $M \boxtimes N$ has a filtration, of length rs, with factors $E_i \boxtimes F_j$ where $1 \le i \le r$ and $1 \le j \le s$.*

Proof. We argue by induction on $r+s$. The case where $r = s = 1$ is clear. Suppose that the first two terms in the filtration of M are $M' \subseteq M$ and $M/M' \cong E_1$. Then $M' \boxtimes N \subseteq M \boxtimes N$ has quotient $(M \boxtimes N)/(M' \boxtimes N) \cong E_1 \boxtimes N$. By the natural correspondence between the submodules of $(M\boxtimes N)/(M'\boxtimes N)$ and those of $M\boxtimes N$ containing $M'\boxtimes N$, from the filtration of N, we obtain a filtration

$$M' \boxtimes N = T_{s+1} \subseteq T_s \subseteq \cdots \subseteq T_2 \subseteq T_1 = M \boxtimes N$$

such that $T_j/T_{j+1} \cong E_1 \boxtimes F_j$ for $j = 1,\ldots,s$. The proof is now complete by induction. □

We are now ready to prove the signed Young's rule when the composition α in the bicomposition $(\alpha|\beta)$ is empty.

Lemma 3.3. *Let β be a composition of n. Then $M_\mathbb{F}(\varnothing|\beta)$ has a Specht filtration. For a partition λ of n, the multiplicity of $S^\lambda_\mathbb{F}$ as a factor of this filtration equals $|\mathscr{T}_{\mathrm{sstd}}(\lambda',\beta)|$.*

Proof. In our proof, we omit the subscript $\mathbb{F}$ throughout. By Theorem 2.2, M^β has a Specht filtration

$$M^\beta = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = 0$$

in which S^λ occurs with multiplicity $|\mathscr{T}_{\mathrm{sstd}}(\lambda,\beta)|$. Since M^β is self-dual and $M(\varnothing|\beta) \cong M(\beta|\varnothing) \otimes \mathrm{sgn} = M^\beta \otimes \mathrm{sgn}$, by Lemma 3.1, we have the following filtration for

$M(\varnothing|\beta)$:

$$0 = M_0^\perp \otimes \mathrm{sgn} \subseteq M_1^\perp \otimes \mathrm{sgn} \subseteq \cdots \subseteq M_r^\perp \otimes \mathrm{sgn} \cong M^\beta \otimes \mathrm{sgn} \cong M(\varnothing|\beta).$$

Observe that

$$\frac{M_i^\perp \otimes \mathrm{sgn}}{M_{i-1}^\perp \otimes \mathrm{sgn}} \cong \frac{M_i^\perp}{M_{i-1}^\perp} \otimes \mathrm{sgn} \cong \left(\frac{M_{i-1}}{M_i}\right)^* \otimes \mathrm{sgn}.$$

Thus if $\frac{M_{i-1}}{M_i} \cong S^\lambda$, then $\frac{M_i^\perp \otimes \mathrm{sgn}}{M_{i-1}^\perp \otimes \mathrm{sgn}} \cong (S^\lambda)^* \otimes \mathrm{sgn} \cong S^{\lambda'}$. Hence, $S^{\lambda'}$ occurs as a factor of the filtration of $M(\varnothing|\beta)$ with multiplicity $\mathscr{T}_{\mathrm{sstd}}(\lambda,\beta)$, and the lemma follows. □

To deal with the general case, we need the following proposition.

Proposition 3.4. *Let λ be a partition of n, and $(\alpha|\beta)$ be a bicomposition of n. Then*

$$|\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))| = \sum_{\xi,\eta} |\mathscr{T}_{\mathrm{sstd}}(\xi,\alpha)||\mathscr{T}_{\mathrm{sstd}}(\eta',\beta)|\, c^{\lambda}_{\xi\eta},$$

where ξ and η run over all partitions of $|\alpha|$ and $|\beta|$ respectively.

Proof. For each $\mathrm{T} \in \mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))$, let α_{T} be the subpartition of λ whose Young diagram $[\alpha_{\mathrm{T}}] = \{(i,j) \in [\lambda] : \mathrm{T}(i,j) = \mathbb{c}_k \text{ for some } k\}$. Then α_{T}-tableau $\mathrm{T}|_{[\alpha_{\mathrm{T}}]}$ of type α obtained from T is standard, i.e. $\mathrm{T}|_{[\alpha_{\mathrm{T}}]} \in \mathscr{T}_{\mathrm{sstd}}(\alpha_{\mathrm{T}},\alpha)$. Let $\mathrm{T}^\beta : [\lambda'/(\alpha_{\mathrm{T}})'] \to \{\mathbb{c}_1,\dots,\mathbb{c}_{\ell(\beta)}\}$ be defined by $\mathrm{T}^\beta(i,j) = \mathbb{c}_k$ if and only if $\mathrm{T}(j,i) = \mathbb{d}_k$. Then T^β is a semistandard skew $\lambda'/(\alpha_{\mathrm{T}})'$-tableau of type β.

Thus we have a function $f : \mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta)) \to \bigcup_\xi (\mathscr{T}_{\mathrm{sstd}}(\xi,\alpha) \times \mathscr{T}_{\mathrm{sstd}}(\lambda'/\xi',\beta))$ where $f(\mathrm{T}) = (\mathrm{T}|_{[\alpha_{\mathrm{T}}]},\mathrm{T}^\beta)$ and with ξ running over all partitions of $|\alpha|$ which are subpartitions of λ. In fact, it is easy to see that f is a bijection. Consequently,

$$|\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))| = \sum_{\xi} |\mathscr{T}_{\mathrm{sstd}}(\xi,\alpha)||\mathscr{T}_{\mathrm{sstd}}(\lambda'/\xi',\beta)|,$$

with ξ running over all partitions of $|\alpha|$ which are subpartitions of λ.

By [F, §5.1, Corollary 2(iii)], when β is a partition, we have

$$|\mathscr{T}_{\mathrm{sstd}}(\lambda'/\xi',\beta)| = \sum_{\eta} |\mathscr{T}_{\mathrm{sstd}}(\eta',\beta)| c^{\lambda}_{\xi,\eta},$$

where η runs over all partitions of $|\beta|$. When β is a composition, with the partition obtained by rearranging its parts denoted by $\bar{\beta}$, we have, by Corollary 2.3,

$$|\mathscr{T}_{\mathrm{sstd}}(\lambda'/\xi',\beta)| = |\mathscr{T}_{\mathrm{sstd}}(\lambda'/\xi',\bar{\beta})|,$$
$$|\mathscr{T}_{\mathrm{sstd}}(\eta',\beta)| = |\mathscr{T}_{\mathrm{sstd}}(\eta',\bar{\beta})|.$$

Hence,

$$\begin{aligned}|\mathscr{T}_{\mathrm{sstd}}(\lambda'/\xi',\beta)| &= |\mathscr{T}_{\mathrm{sstd}}(\lambda'/\xi',\bar{\beta})| \\ &= \sum_{\eta} |\mathscr{T}_{\mathrm{sstd}}(\eta',\bar{\beta})| c^{\lambda}_{\xi,\eta} \\ &= \sum_{\eta} |\mathscr{T}_{\mathrm{sstd}}(\eta',\beta)| c^{\lambda}_{\xi,\eta}.\end{aligned}$$

Now, since $c^{\lambda}_{\xi,\eta} = 0$ unless ξ and η are subpartitions of λ, the proposition follows immediately. □

We are now ready to prove the main theorem of this section.

Theorem 3.5 (Signed Young's rule)**.** *Let $(\alpha|\beta)$ be a bicomposition of n. The signed Young permutation module $M_{\mathbb{F}}(\alpha|\beta)$ has a Specht filtration in which, for a partition λ of n, the multiplicity of $S^{\lambda}_{\mathbb{F}}$ as a factor of the filtration equals $|\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))|$.*

Dually, the signed Young permutation module $M_{\mathbb{F}}(\alpha|\beta)$ has a dual Specht filtration in which, for a partition λ of n, the multiplicity of $S_{\lambda,\mathbb{F}} \cong (S^{\lambda}_{\mathbb{F}})^*$ as a factor of the filtration equals $|\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))|$.

Proof. In our proof, we omit the subscript $\mathbb{F}$. Observe first that

$$\begin{aligned}M(\alpha|\beta) = \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\alpha\times\mathfrak{S}_\beta}(\mathbb{F}\boxtimes \mathrm{sgn}) &\cong \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{|\alpha|}\times\mathfrak{S}_{|\beta|}}\left(\left(\mathrm{Ind}^{\mathfrak{S}_{|\alpha|}}_{\mathfrak{S}_\alpha}\mathbb{F}\right)\boxtimes\left(\mathrm{Ind}^{\mathfrak{S}_{|\beta|}}_{\mathfrak{S}_\beta}\mathrm{sgn}\right)\right) \\ &\cong \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{|\alpha|}\times\mathfrak{S}_{|\beta|}}(M^\alpha\boxtimes M(\varnothing|\beta)).\end{aligned}$$

By Theorem 2.2 and Lemmas 3.2 and 3.3, $M^\alpha\boxtimes M(\varnothing|\beta)$ has a filtration in which every factor has the form $S^\xi\boxtimes S^\eta$ for some partitions ξ of $|\alpha|$ and η of $|\beta|$, and the multiplicity of $S^\xi\boxtimes S^\eta$ as a factor of the filtration equals $|\mathscr{T}_{\mathrm{sstd}}(\xi,\alpha)||\mathscr{T}_{\mathrm{sstd}}(\eta',\beta)|$.

As the induction functor is exact, $M(\alpha|\beta)$ thus has a filtration in which every factor has the form $\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{|\alpha|}\times\mathfrak{S}_{|\beta|}}(S^\xi\boxtimes S^\eta)$ for some partitions ξ of $|\alpha|$ and η of $|\beta|$, and the multiplicity of $\mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{|\alpha|}\times\mathfrak{S}_{|\beta|}}(S^\xi\boxtimes S^\eta)$ as a factor of the filtration equals $|\mathscr{T}_{\mathrm{sstd}}(\xi,\alpha)||\mathscr{T}_{\mathrm{sstd}}(\eta',\beta)|$.

By Theorem 2.4, this filtration can be further refined to a Specht filtration in which the multiplicity of S^λ as a factor equals $|\mathscr{T}_{\mathrm{sstd}}(\xi,\alpha)||\mathscr{T}_{\mathrm{sstd}}(\eta',\beta)|c^{\lambda}_{\xi\eta}$. Proposition 3.4 now completes the proof of the first assertion.

The second assertion is obtained by applying Lemma 3.1 to the Specht filtration we have just obtained, together with the fact that $M(\alpha|\beta)$ is self-dual. □

As a consequence, we obtain the following interesting combinatorial equalities.

Corollary 3.6. *Let $(\alpha|\beta)$ be a bicomposition of n.*

(i) $[\mathfrak{S}_n:\mathfrak{S}_{\alpha|\beta}] = \sum_\lambda |\mathscr{T}_{\mathrm{std}}(\lambda)||\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))|$, *where the sum runs over all partitions λ of n.*

(ii) *For any partition λ of n, $|\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))| = |\mathscr{T}_{\mathrm{sstd}}(\lambda',(\beta|\alpha))|$.*

(iii) *Let $\tilde{\alpha}$, $\tilde{\beta}$ be compositions obtained by rearranging some parts of α and β respectively. For any partition λ of n, $|\mathscr{T}_{\mathrm{sstd}}(\lambda,(\tilde{\alpha}|\tilde{\beta}))| = |\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))|$.*

(iv) *For any partition λ of $m+n$, $|\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha\sqcup(1^m)|\beta))| = |\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta\sqcup(1^m)))|$ where $\sqcup$ denotes the concatenation of compositions.*

Proof.

(i) It is well known that the Specht module $S^\lambda_{\mathbb{F}}$ has dimension $|\mathscr{T}_{\mathrm{std}}(\lambda)|$ (see, for example, [P2, Theorem 1.1]). Taking the dimension of $M(\alpha|\beta)$ and applying Theorem 3.5 thus yield (i).

(ii) When $\mathbb{F}=\mathbb{Q}$, the Specht modules are the irreducible modules, so that the Specht filtration in Theorem 3.5 is in fact a composition series. Thus

$$\begin{aligned}|\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))| = [M_{\mathbb{Q}}(\alpha|\beta):S^\lambda_{\mathbb{Q}}] &= [M_{\mathbb{Q}}(\alpha|\beta)\otimes \mathrm{sgn} : S^\lambda_{\mathbb{Q}}\otimes\mathrm{sgn}]\\ &= [M_{\mathbb{Q}}(\beta|\alpha):S^{\lambda'}_{\mathbb{Q}}] = |\mathscr{T}_{\mathrm{sstd}}(\lambda',(\beta|\alpha))|.\end{aligned}$$

(iii,iv) These are proved in a similar but easier manner as part (ii), using the isomorphisms $M(\alpha|\beta)\cong M(\tilde\alpha|\tilde\beta)$ and $M(\alpha\sqcup(1^m)|\beta)\cong M(\alpha|\beta\sqcup(1^m))$ respectively. □

4. Homomorphisms between Specht modules and signed Young permutation modules

Let λ be a partition of n and μ be a composition of n. In [J, §13], James constructed for each $\mathrm{T}\in\mathscr{T}(\lambda,\mu)$ an $\mathbb{F}\mathfrak{S}_n$-module homomorphism $\theta_{\mathrm{T}}: M^\lambda_{\mathbb{F}}\to M^\mu_{\mathbb{F}}$, and showed that $\{\hat\theta_{\mathrm{T}} := \theta_{\mathrm{T}}|_{S^\lambda_{\mathbb{F}}} : \mathrm{T}\in\mathscr{T}_{\mathrm{sstd}}(\lambda,\mu)\}$ is a basis for the space $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M^\mu_{\mathbb{F}})$, unless $\mathbb{F}$ has characteristic 2 and λ is 2-singular. In particular, this shows that every homomorphism in $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M^\mu_{\mathbb{F}})$ is the restriction of a homomorphism in $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(M^\lambda_{\mathbb{F}}, M^\mu_{\mathbb{F}})$ when the characteristic of $\mathbb{F}$ is not 2.

In this section, we shall generalise these homomorphisms to obtain homomorphisms between Specht modules and signed Young permutation modules. The next example shows that not every homomorphism in $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$ is the restriction of a homomorphism in $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(M^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$, illustrating the difficulty of such generalisation.

Example 4.1. Let $n\in\mathbb{Z}^+$ and let $\mathbb{F}$ have positive characteristic p with $p\le n$. Let $\lambda=(1^n)$ and $(\alpha|\beta)=(\varnothing|(n))$. Then $M^\lambda_{\mathbb{F}}$ is the regular $\mathbb{F}\mathfrak{S}_n$-module $\mathbb{F}\mathfrak{S}_n$, while both $M_{\mathbb{F}}(\alpha|\beta)=M_{\mathbb{F}}(\varnothing|(n))$ and $S^\lambda_{\mathbb{F}}$ are isomorphic to the signature representation sgn of $\mathbb{F}\mathfrak{S}_n$. Thus $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$ has dimension one.

On the other hand, $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(M^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))\cong\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}\mathfrak{S}_n,\mathrm{sgn})$ has dimension 1 with a basis $\{\theta:\mathbb{F}\mathfrak{S}_n\to\mathrm{sgn}\}$, where $\theta(1_{\mathfrak{S}_n})=\epsilon$. For any $\sigma\in\mathfrak{S}_n$, we have

$$\theta(\sigma)=\sigma\cdot\theta(1_{\mathfrak{S}_n})=\mathrm{sgn}(\sigma)\epsilon.$$

As a submodule of $M^\lambda_{\mathbb{F}}=\mathbb{F}\mathfrak{S}_n$, the Specht module $S^\lambda_{\mathbb{F}}$ ($\cong$ sgn) is generated by $\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\sigma$. We have

$$\theta\left(\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\sigma\right)=\sum_{\sigma\in\mathfrak{S}_n}\mathrm{sgn}(\sigma)\theta(\sigma)=\sum_{\sigma\in\mathfrak{S}_n}\epsilon=(n!)\epsilon=0,$$

since $p\le n$. Thus $\theta|_{S^\lambda_{\mathbb{F}}}=0$.

This example shows that the map $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(M^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta)) \to \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$ defined by $\theta \mapsto \theta|_{S^\lambda_{\mathbb{F}}}$ is not surjective in general.

As such, to generalise the homomorphisms constructed by James for signed Young permutation modules, we should not attempt to generalise θ_{T} and take its restriction to $S^\lambda_{\mathbb{F}}$, but have to generalise $\hat{\theta}_{\mathrm{T}}$ directly instead. In other words, we need to understand $\hat{\theta}_{\mathrm{T}}(e_{\mathfrak{t}})$.

4.1. **James's construction.** Let λ be a partition of n and μ be a composition of n. In [J, §13], James defined an $\mathbb{F}\mathfrak{S}_n$-module homomorphism $\theta_{\mathrm{T}} : M^\lambda_{\mathbb{F}} \to M^\mu_{\mathbb{F}}$ for each $\mathrm{T} \in \mathscr{T}(\lambda,\mu)$. In this subsection, we study how θ_{T} acts on the polytabloids in $M^\lambda_{\mathbb{F}}$.

Fix a λ-tableau $\mathfrak{t}_0$, so that $\mathfrak{S}_n$ acts on $\mathscr{T}(\lambda,\mu)$ via $\mathfrak{t}_0$, as described in Subsection 2.4. Let T_0 be the canonical λ-tableau of type μ associated to $\mathfrak{t}_0$; recall that $\mathrm{stab}_{\mathfrak{S}_n}(\mathrm{T}_0) = \mathfrak{S}_\mu$. Then each left coset of $\mathfrak{S}_\mu$ in $\mathfrak{S}_n$ corresponds to a λ-tableau of type μ; let $d_{\mathrm{T}}\mathfrak{S}_\mu$ be the left coset corresponding to T. Recall also the initial λ-tableau $\mathfrak{t}^\lambda$, and let $d_{\mathfrak{t}_0} = \mathfrak{t}_0 \circ (\mathfrak{t}^\lambda)^{-1} \in \mathfrak{S}_n$. James defined $\theta_{\mathrm{T}} \in \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(M^\lambda_{\mathbb{F}}, M^\mu_{\mathbb{F}})$ so that

$$\theta_{\mathrm{T}}(d_{\mathfrak{t}_0}\mathfrak{S}_\lambda) = \sum_{\substack{d\mathfrak{S}_\mu \in \mathfrak{S}_n/\mathfrak{S}_\mu: \\ d\mathfrak{S}_\mu \subseteq R_{\mathfrak{t}_0} d_{\mathrm{T}}\mathfrak{S}_\mu}} d\mathfrak{S}_\mu.$$

Let $\mathfrak{t} \in \mathscr{T}(\lambda)$, and let $\rho_{\mathfrak{t}} = \mathfrak{t} \circ (\mathfrak{t}_0)^{-1}$. Then $\rho_{\mathfrak{t}} \cdot \mathfrak{t}_0 = \rho_{\mathfrak{t}} \circ \mathfrak{t}_0 = \mathfrak{t}$, so that $e_{\mathfrak{t}} = e_{\rho_{\mathfrak{t}} \cdot \mathfrak{t}_0} = \rho_{\mathfrak{t}} \cdot e_{\mathfrak{t}_0}$. Thus

$$\begin{aligned}
\theta_{\mathrm{T}}(e_{\mathfrak{t}}) &= \theta_{\mathrm{T}}(\rho_{\mathfrak{t}} \cdot e_{\mathfrak{t}_0}) \\
&= \theta_{\mathrm{T}}\left(\rho_{\mathfrak{t}} \cdot \sum_{\sigma \in C_{\mathfrak{t}_0}} \mathrm{sgn}(\sigma)(\sigma \cdot (d_{\mathfrak{t}_0}\mathfrak{S}_\lambda))\right) \\
&= \sum_{\sigma \in C_{\mathfrak{t}_0}} \mathrm{sgn}(\sigma)(\rho_{\mathfrak{t}}\sigma \cdot \theta_{\mathrm{T}}(d_{\mathfrak{t}_0}\mathfrak{S}_\lambda)) \\
&= \sum_{\sigma \in C_{\mathfrak{t}_0}} \mathrm{sgn}(\sigma)\left(\rho_{\mathfrak{t}}\sigma \cdot \sum_{\substack{d\mathfrak{S}_\mu \in \mathfrak{S}_n/\mathfrak{S}_\mu: \\ d\mathfrak{S}_\mu \subseteq R_{\mathfrak{t}_0} d_{\mathrm{T}}\mathfrak{S}_\mu}} d\mathfrak{S}_\mu\right) \\
&= \sum_{\sigma \in C_{\mathfrak{t}_0}} \mathrm{sgn}(\sigma) \sum_{\substack{d\mathfrak{S}_\mu \in \mathfrak{S}_n/\mathfrak{S}_\mu: \\ d\mathfrak{S}_\mu \subseteq \rho_{\mathfrak{t}}\sigma R_{\mathfrak{t}_0} d_{\mathrm{T}}\mathfrak{S}_\mu}} d\mathfrak{S}_\mu \\
&= \sum_{d\mathfrak{S}_\mu \in \mathfrak{S}_n/\mathfrak{S}_\mu} a_{\rho_{\mathfrak{t}}^{-1}d, d_{\mathrm{T}}}\, d\mathfrak{S}_\mu;
\end{aligned}$$

where

$$a_{\rho_{\mathfrak{t}}^{-1}d, d_{\mathrm{T}}} = \sum_{\substack{\sigma \in C_{\mathfrak{t}_0}: \\ \sigma^{-1}\rho_{\mathfrak{t}}^{-1}d \in R_{\mathfrak{t}_0} d_{\mathrm{T}}\mathfrak{S}_\mu}} \mathrm{sgn}(\sigma) = \sum_{\substack{\sigma \in C_{\mathfrak{t}_0}: \\ \sigma\rho_{\mathfrak{t}}^{-1}d \in R_{\mathfrak{t}_0} d_{\mathrm{T}}\mathfrak{S}_\mu}} \mathrm{sgn}(\sigma).$$

We summarise this below.

Theorem 4.2. *Let λ be a partition of n and μ be a composition of n, and let $\mathrm{T} \in \mathscr{T}(\lambda,\mu)$. The $\mathbb{F}\mathfrak{S}_n$-module homomorphism $\hat{\theta}_{\mathrm{T}} : S^\lambda_{\mathbb{F}} \to M^\mu_{\mathbb{F}}$ constructed by James satisfies $\hat{\theta}_{\mathrm{T}}(e_{\mathfrak{t}}) = \sum_{d\mathfrak{S}_\mu \in \mathfrak{S}_n/\mathfrak{S}_\mu} a_{\rho_{\mathfrak{t}}^{-1}d,d_{\mathrm{T}}}\, d\mathfrak{S}_\mu$ where*

$$a_{\rho_{\mathfrak{t}}^{-1}d,d_{\mathrm{T}}} = \sum_{\substack{\sigma\in C_{\mathfrak{t}_0}:\\ \sigma\rho_{\mathfrak{t}}^{-1}d\in R_{\mathfrak{t}_0}d_{\mathrm{T}}\mathfrak{S}_\mu}} \mathrm{sgn}(\sigma).$$

4.2. **Generalization of James's construction.** Fix a partition λ of n and a bicomposition $(\alpha|\beta)$ of n. In this subsection, we generalise James's construction of homomorphisms between Specht modules and Young permutation modules to obtain, for each d in a subset $\mathscr{R}$ of $\mathfrak{S}_n$ to be defined below, a $\mathbb{Z}\mathfrak{S}_n$-module homomorphism $\bar{\vartheta}_d : S^\lambda_{\mathbb{Z}} \to M_{\mathbb{Z}}(\alpha|\beta)$ (see Theorem 4.15).

As before, we fix a λ-tableau $\mathfrak{t}_0$, so that $\mathfrak{S}_n$ acts on $\mathscr{T}(\lambda,(\alpha|\beta))$ through $\mathfrak{t}_0$, and denote the canonical λ-tableau of type $(\alpha|\beta)$ associated to $\mathfrak{t}_0$ by T_0.

Definition 4.3. Define subsets of $\mathfrak{S}_n$ as follows:

$$\mathscr{R} = \{d \in \mathfrak{S}_n : d^{-1}R_{\mathfrak{t}_0}d \cap \mathfrak{S}_{\alpha|\beta} \subseteq \mathfrak{S}_\alpha\};$$
$$\mathscr{C} = \{d \in \mathfrak{S}_n : d^{-1}C_{\mathfrak{t}_0}d \cap \mathfrak{S}_{\alpha|\beta} \subseteq \mathfrak{S}_\beta^{+|\alpha|}\}.$$

These sets $\mathscr{R}$ and $\mathscr{C}$ of course depend on $\mathfrak{t}_0$ and $(\alpha|\beta)$.

Lemma 4.4. *Let $d \in \mathfrak{S}_n$.*

(i) *The following statements are equivalent:*
  (a) *$d \in \mathscr{R}$.*
  (b) $\mathrm{stab}_{R_{\mathfrak{t}_0}}(\mathrm{T}_d) \subseteq d\mathfrak{S}_\alpha d^{-1}$.
  (c) *Whenever $\mathrm{T}_d(i,j) = \mathrm{T}_d(i,j')$ for some $(i,j),(i,j') \in [\lambda]$ with $j \neq j'$, we have $\mathrm{T}_d(i,j) = \mathbb{c}_k$ for some k.*

(ii) *If $d \in \mathscr{R}$, then $\tau d\xi \in \mathscr{R}$ for all $\tau \in R_{\mathfrak{t}_0}$ and $\xi \in \mathfrak{S}_{\alpha|\beta}$.*

Proof. For part (i), firstly,

$$\mathrm{stab}_{R_{\mathfrak{t}_0}}(\mathrm{T}_d) = R_{\mathfrak{t}_0} \cap \mathrm{stab}_{\mathfrak{S}_n}(d\cdot \mathrm{T}_0) = R_{\mathfrak{t}_0} \cap d\mathfrak{S}_{\alpha|\beta}d^{-1}.$$

This proves the equivalence of (a) and (b).

Next, observe that a transposition $(a\ \ b)$ lies in $R_{\mathfrak{t}_0}$ if and only if a and b label nodes in the same row of $\mathfrak{t}_0$, i.e. $(\mathfrak{t}_0)^{-1}(a) = (i,j)$ and $(\mathfrak{t}_0)^{-1}(b) = (i,j')$ for some i. Furthermore, $(a\ \ b)\cdot \mathrm{T}_d = \mathrm{T}_d$ if and only if $\mathrm{T}_d((\mathfrak{t}_0)^{-1}(a)) = \mathrm{T}_d((\mathfrak{t}_0)^{-1}(b))$. Since $\mathrm{stab}_{R_{\mathfrak{t}_0}}(\mathrm{T}_d) = R_{\mathfrak{t}_0} \cap d\mathfrak{S}_{\alpha|\beta}d^{-1}$ is an intersection of conjugates of Young subgroups, it is generated by the transpositions it contains. Thus, $\mathrm{stab}_{R_{\mathfrak{t}_0}}(\mathrm{T}_d)$ is generated by

$$S = \{(\mathfrak{t}_0(i,j)\ \ \mathfrak{t}_0(i,j')) : (i,j),(i,j') \in [\lambda],\ j \neq j',\ \mathrm{T}_d(i,j) = \mathrm{T}_d(i,j')\}.$$

Now,

$$\begin{aligned}
&(\mathfrak{t}_0(i,j)\ \ \mathfrak{t}_0(i,j')) \in d\mathfrak{S}_\alpha d^{-1} \\
&\Leftrightarrow (d^{-1}(\mathfrak{t}_0(i,j))\ \ d^{-1}(\mathfrak{t}_0(i,j'))) \in \mathfrak{S}_\alpha \\
&\Leftrightarrow \exists(1 \le k \le \ell(\alpha)), \sum_{i=1}^{k-1} \alpha_i < d^{-1}(\mathfrak{t}_0(i,j)), d^{-1}(\mathfrak{t}_0(i,j')) \le \sum_{i=1}^{k} \alpha_i \\
&\Leftrightarrow \exists(1 \le k \le \ell(\alpha)), \mathrm{T}_0((\mathfrak{t}_0)^{-1}(d^{-1}(\mathfrak{t}_0(i,j)))) = \mathrm{T}_0((\mathfrak{t}_0)^{-1}(d^{-1}(\mathfrak{t}_0(i,j')))) = \mathbb{c}_k \\
&\Leftrightarrow \exists(1 \le k \le \ell(\alpha)), \mathrm{T}_d(i,j) = \mathrm{T}_d(i,j') = \mathbb{c}_k.
\end{aligned}$$

Hence (b) and (c) are equivalent.

For part (ii), observe that

$$(\tau d\xi)^{-1} R_{\mathfrak{t}_0} (\tau d\xi) \cap \mathfrak{S}_{\alpha|\beta} = \xi^{-1}(d^{-1} R_{\mathfrak{t}_0} d \cap \mathfrak{S}_{\alpha|\beta})\xi \subseteq \xi^{-1}\mathfrak{S}_\alpha \xi = \mathfrak{S}_\alpha.$$

□

We have analogous statements and proofs for $\mathscr{C}$ too.

Lemma 4.5. *Let $d \in \mathfrak{S}_n$.*

(i) *The following statements are equivalent:*
  (a) *$d \in \mathscr{C}$.*
  (b) $\mathrm{stab}_{C_{\mathfrak{t}_0}}(\mathrm{T}_d) \subseteq d\mathfrak{S}_\beta^{+|\alpha|} d^{-1}$.
  (c) *Whenever $\mathrm{T}_d(i,j) = \mathrm{T}_d(i',j)$ for some $(i,j),(i',j) \in [\lambda]$ with $i \neq i'$, we have $\mathrm{T}_d(i,j) = \mathbb{d}_k$ for some k.*

(ii) *If $d \in \mathscr{C}$, then $\tau d\xi \in \mathscr{C}$ for all $\tau \in C_{\mathfrak{t}_0}$ and $\xi \in \mathfrak{S}_{\alpha|\beta}$.*

Lemmas 4.4 and 4.5 give the following immediate corollary.

Corollary 4.6. *If $d \in \mathfrak{S}_n$ such that $\mathrm{T}_d \in \mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))$, then $d \in \mathscr{R} \cap \mathscr{C}$.*

In order to generalise the coefficient a_{d,d_T} in Theorem 4.2 to $\mathfrak{a}_{d,\mathfrak{d}}$ in Definition 4.8, we need the following lemma which also explains the choice of the set $\mathscr{R}$.

Lemma 4.7. *Let $\mathfrak{d} \in \mathscr{R}$. There is a well-defined projection map $\pi_\mathfrak{d} : R_{\mathfrak{t}_0}\mathfrak{d}\mathfrak{S}_{\alpha|\beta} \to \mathfrak{S}_\beta$ defined by $\tau\mathfrak{d}\xi_\alpha\xi_\beta^{+|\alpha|} \mapsto \xi_\beta$ for all $\tau \in R_{\mathfrak{t}_0}$ and $(\xi_\alpha,\xi_\beta) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$.*

Proof. If $\tau\mathfrak{d}\xi_\alpha\xi_\beta^{+|\alpha|} = \tau'\mathfrak{d}\xi'_\alpha\xi_\beta'^{+|\alpha|}$, then

$$\mathfrak{d}^{-1} R_{\mathfrak{t}_0}\mathfrak{d} \ni \mathfrak{d}^{-1}(\tau^{-1}\tau')\mathfrak{d} = \xi_\alpha\xi_\beta^{+|\alpha|}(\xi'_\alpha\xi_\beta'^{+|\alpha|})^{-1} \in \mathfrak{S}_{\alpha|\beta},$$

so that $\xi_\alpha\xi_\beta^{+|\alpha|}(\xi'_\alpha\xi_\beta'^{+|\alpha|})^{-1} \in \mathfrak{S}_\alpha$ since $\mathfrak{d} \in \mathscr{R}$, forcing $\xi_\beta = \xi'_\beta$. The lemma thus follows. □

In view of Lemma 4.7, we can make the following definition.

Definition 4.8. Let $d, \mathfrak{d} \in \mathfrak{S}_n$ with $\mathfrak{d} \in \mathscr{R}$.

(i) For $\omega \in R_{\mathfrak{t}_0}\mathfrak{d}\mathfrak{S}_{\alpha|\beta}$, let $\varepsilon_\mathfrak{d}(\omega) := \mathrm{sgn}(\pi_\mathfrak{d}(\omega)) \in \{\pm 1\}$.

(ii) Let

$$\Omega_{d,\mathfrak{d}} := \{\sigma \in C_{\mathfrak{t}_0} : \sigma d \in R_{\mathfrak{t}_0}\mathfrak{d}\mathfrak{S}_{\alpha|\beta}\},$$
$$\mathfrak{a}_{d,\mathfrak{d}} := \sum_{\sigma\in\Omega_{d,\mathfrak{d}}} \operatorname{sgn}(\sigma)\varepsilon_{\mathfrak{d}}(\sigma d) \in \mathbb{Z}.$$

By convention, if $\Omega_{d,\mathfrak{d}} = \emptyset$ then $\mathfrak{a}_{d,\mathfrak{d}} = 0$.

Remark 4.9. We give another description of $\Omega_{d,\mathfrak{d}}$ here. Restrict the left regular action of the symmetric group $\mathfrak{S}_n$ on $\mathfrak{S}_n/\mathfrak{S}_{\alpha|\beta}$ to the subgroups $R_{\mathfrak{t}_0}$ and $C_{\mathfrak{t}_0}$, which partition $\mathfrak{S}_n/\mathfrak{S}_{\alpha|\beta}$ into $R_{\mathfrak{t}_0}$-orbits and into $C_{\mathfrak{t}_0}$-orbits respectively. Then $\sigma \in \Omega_{d,\mathfrak{d}}$ if and only if $\sigma \in C_{\mathfrak{t}_0}$ and $\sigma \cdot (d\mathfrak{S}_{\alpha|\beta}) \in R_{\mathfrak{t}_0} \cdot (\mathfrak{d}\mathfrak{S}_{\alpha|\beta})$. As such, $\Omega_{d,\mathfrak{d}} \neq \emptyset$ if and only if $C_{\mathfrak{t}_0} \cdot (d\mathfrak{S}_{\alpha|\beta}) \cap R_{\mathfrak{t}_0} \cdot (\mathfrak{d}\mathfrak{S}_{\alpha|\beta}) \neq \emptyset$. Furthermore, if $C_{\mathfrak{t}_0} \cdot (d\mathfrak{S}_{\alpha|\beta}) \cap R_{\mathfrak{t}_0} \cdot (\mathfrak{d}\mathfrak{S}_{\alpha|\beta})$ contains precisely the distinct left cosets $d^{(1)}\mathfrak{S}_{\alpha|\beta}, d^{(2)}\mathfrak{S}_{\alpha|\beta}, \dots, d^{(r)}\mathfrak{S}_{\alpha|\beta}$, then $\Omega_{d,\mathfrak{d}} = \bigcup_{i=1}^{r} \Omega_i$ (disjoint union), where for each i, $\Omega_i = \{\sigma \in C_{\mathfrak{t}_0} : \sigma \cdot d\mathfrak{S}_{\alpha|\beta} = d^{(i)}\mathfrak{S}_{\alpha|\beta}\}$ and is therefore a left coset of $\operatorname{stab}_{C_{\mathfrak{t}_0}}(d\mathfrak{S}_{\alpha|\beta}) = C_{\mathfrak{t}_0} \cap d\mathfrak{S}_{\alpha|\beta}d^{-1}$. In particular, $\Omega_{d,\mathfrak{d}}$ is a union of some left cosets of $C_{\mathfrak{t}_0} \cap d\mathfrak{S}_{\alpha|\beta}d^{-1}$.

We collect together some important properties that $\mathfrak{a}_{d,\mathfrak{d}}$ satisfies:

Lemma 4.10. *Let $d, \mathfrak{d} \in \mathfrak{S}_n$ with $\mathfrak{d} \in \mathscr{R}$.*

(i) *If $\mathfrak{d}' = \tau\mathfrak{d}\xi_\alpha\xi_\beta^{+|\alpha|}$ and $d' = \sigma d\eta_\alpha\eta_\beta^{+|\alpha|}$ for some $\tau \in R_{\mathfrak{t}_0}$, $\sigma \in C_{\mathfrak{t}_0}$ and $(\xi_\alpha, \xi_\beta), (\eta_\alpha, \eta_\beta) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$, then $\mathfrak{d}' \in \mathscr{R}$ and*
$$\mathfrak{a}_{d',\mathfrak{d}'} = \operatorname{sgn}(\sigma)\operatorname{sgn}(\xi_\beta)\operatorname{sgn}(\eta_\beta)\mathfrak{a}_{d,\mathfrak{d}}.$$

(ii) *If $C_{\mathfrak{t}_0}d\mathfrak{S}_{\alpha|\beta} \cap R_{\mathfrak{t}_0}\mathfrak{d}\mathfrak{S}_{\alpha|\beta} = \emptyset$ or $d \notin \mathscr{C}$, then $\mathfrak{a}_{d,\mathfrak{d}} = 0$.*

(iii) *Suppose that $d \in \mathscr{C}$ and that $C_{\mathfrak{t}_0}d\mathfrak{S}_{\alpha|\beta} \cap R_{\mathfrak{t}_0}\mathfrak{d}\mathfrak{S}_{\alpha|\beta}$ is a disjoint union of r left cosets of $\mathfrak{S}_{\alpha|\beta}$, with representatives $d^{(1)}, \dots, d^{(r)}$. For each i, let $\sigma_i \in C_{\mathfrak{t}_0}$ such that $\sigma_i d \in d^{(i)}\mathfrak{S}_{\alpha|\beta}$, and let $\varepsilon^{(i)} = \operatorname{sgn}(\sigma_i)\varepsilon_{\mathfrak{d}}(\sigma_i d)$. Then*
$$\mathfrak{a}_{d,\mathfrak{d}} = |C_{\mathfrak{t}_0} \cap d\mathfrak{S}_{\alpha|\beta}d^{-1}| \sum_{i=1}^{r} \varepsilon^{(i)}.$$

Proof. By Lemma 4.4(ii), $\mathfrak{d}' \in \mathscr{R}$. It is also easy to see that
$$\Omega_{d',\mathfrak{d}'} = \Omega_{\sigma d,\mathfrak{d}} = \Omega_{d,\mathfrak{d}}\sigma^{-1}.$$
Take $\omega \in \Omega_{d',\mathfrak{d}'}$, say $\omega d' = \tau'\mathfrak{d}'\gamma_\alpha\gamma_\beta^{+|\alpha|}$ with $\tau' \in R_{\mathfrak{t}_0}$ and $(\gamma_\alpha, \gamma_\beta) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$. Then
$$\omega\sigma d = \tau'\tau\mathfrak{d}\xi_\alpha\gamma_\alpha\eta_\alpha^{-1}(\xi_\beta\gamma_\beta\eta_\beta^{-1})^{+|\alpha|}$$
and hence we have $\varepsilon_{\mathfrak{d}'}(\omega d') = \operatorname{sgn}(\gamma_\beta) = \varepsilon_{\mathfrak{d}}(\omega\sigma d)\operatorname{sgn}(\eta_\beta)\operatorname{sgn}(\xi_\beta)$. Therefore
$$\begin{aligned}\mathfrak{a}_{d',\mathfrak{d}'} &= \sum_{\omega\in\Omega_{d',\mathfrak{d}'}} \operatorname{sgn}(\omega)\varepsilon_{\mathfrak{d}'}(\omega d') \\ &= \sum_{\omega\sigma\in\Omega_{d,\mathfrak{d}}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\omega\sigma)\varepsilon_{\mathfrak{d}}(\omega\sigma d)\operatorname{sgn}(\xi_\beta)\operatorname{sgn}(\eta_\beta) \\ &= \operatorname{sgn}(\sigma)\operatorname{sgn}(\xi_\beta)\operatorname{sgn}(\eta_\beta)\mathfrak{a}_{d,\mathfrak{d}}.\end{aligned}$$

This completes the proof of part (i).

For part (ii), it is clear that $\Omega_{d,\mathfrak{d}} = \emptyset$ if $C_{\mathfrak{t}_0} d\mathfrak{S}_{\alpha|\beta} \cap R_{\mathfrak{t}_0}\mathfrak{d}\mathfrak{S}_{\alpha|\beta} = \emptyset$, so that $\mathfrak{a}_{d,\mathfrak{d}} = 0$ in this instance. Next we assume that $d \notin \mathscr{C}$. By Lemma 4.5(a), there exist $(i,j),(i',j) \in [\lambda]$ with $i \neq i'$ such that $\mathrm{T}_d(i,j) = \mathbb{c}_k = \mathrm{T}_d(i',j)$ for some k. Let $\rho = (\mathfrak{t}_0(i,j)\ \ \mathfrak{t}_0(i',j))$. Then $\rho \in C_{\mathfrak{t}_0} \cap d\mathfrak{S}_\alpha d^{-1}$. If $\sigma \in \Omega_{d,\mathfrak{d}}$, say $\sigma d = \tau\mathfrak{d}\gamma_\alpha\gamma_\beta^{+|\alpha|}$ with $\tau \in R_{\mathfrak{t}_0}$ and $(\gamma_\alpha,\gamma_\beta) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$, then

$$R_{\mathfrak{t}_0}\mathfrak{d}\mathfrak{S}_{\alpha|\beta} \ni \tau\mathfrak{d}\gamma_\alpha(d^{-1}\rho d)\gamma_\beta^{+|\alpha|} = \sigma d(d^{-1}\rho d) = (\sigma\rho)d,$$

so that $\sigma\rho \in \Omega_{d,\mathfrak{d}}$, and

$$\mathrm{sgn}(\sigma\rho)\varepsilon_{\mathfrak{d}}(\sigma\rho d) = \mathrm{sgn}(\rho)\mathrm{sgn}(\sigma)\mathrm{sgn}(\gamma_\beta) = -\mathrm{sgn}(\sigma)\varepsilon_{\mathfrak{d}}(\sigma d).$$

As such, the contributions to the sum in $\mathfrak{a}_{d,\mathfrak{d}}$ by σ and $\sigma\rho$ cancel each other out. Consequently, $\mathfrak{a}_{d,\mathfrak{d}} = 0$ and the proof of part (ii) is now complete.

For part (iii), for each i, let $\Omega^{(i)} = \{\sigma \in \Omega_{d,\mathfrak{d}} : \sigma d \in d^{(i)}\mathfrak{S}_{\alpha|\beta}\}$, so that $\Omega_{d,\mathfrak{d}}$ is a disjoint union of the $\Omega^{(i)}$'s. Fix i. There exist $\tau_i \in R_{\mathfrak{t}_0}$ and $(\gamma_\alpha^{(i)},\gamma_\beta^{(i)}) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$ such that $\sigma_i d = \tau_i\mathfrak{d}\gamma_\alpha^{(i)}\big(\gamma_\beta^{(i)}\big)^{+|\alpha|}$. For any $\omega \in \Omega^{(i)}$, we have

$$d^{-1}C_{\mathfrak{t}_0}d \ni d^{-1}(\sigma_i^{-1}\omega)d = (\sigma_i d)^{-1}(\omega d) \in \mathfrak{S}_{\alpha|\beta},$$

so that $d^{-1}(\sigma_i^{-1}\omega)d \in \mathfrak{S}_\beta^{+|\alpha|}$ since $d \in \mathscr{C}$. Thus

$$\omega d = \sigma_i d(d^{-1}\sigma_i^{-1}\omega d) = \tau_i\mathfrak{d}\gamma_\alpha^{(i)}(\gamma_\beta^{(i)})^{+|\alpha|}(d^{-1}\sigma_i^{-1}\omega d),$$

so that $\pi_{\mathfrak{d}}(\omega d) = \pi_{\mathfrak{d}}(\sigma_i d)d^{-1}\sigma_i^{-1}\omega d$ and hence

$$\mathrm{sgn}(\omega)\varepsilon_{\mathfrak{d}}(\omega d) = \mathrm{sgn}(\omega)\mathrm{sgn}(\pi_{\mathfrak{d}}(\sigma_i d)d^{-1}\sigma_i^{-1}\omega d) = \mathrm{sgn}(\sigma_i)\varepsilon_{\mathfrak{d}}(\sigma_i d) = \varepsilon^{(i)}.$$

Consequently,

$$\begin{aligned}
\mathfrak{a}_{d,\mathfrak{d}} &= \sum_{\omega\in\Omega_{d,\mathfrak{d}}} \mathrm{sgn}(\omega)\varepsilon_{\mathfrak{d}}(\omega d)\\
&= \sum_{i=1}^{r}\sum_{\omega\in\Omega^{(i)}} \mathrm{sgn}(\omega)\varepsilon_{\mathfrak{d}}(\omega d)\\
&= \sum_{i=1}^{r}\sum_{\omega\in\Omega^{(i)}} \varepsilon^{(i)}\\
&= \sum_{i=1}^{r} |\Omega^{(i)}|\varepsilon^{(i)}\\
&= |C_{\mathfrak{t}_0} \cap d\mathfrak{S}_{\alpha|\beta}d^{-1}| \sum_{i=1}^{r}\varepsilon^{(i)},
\end{aligned}$$

where the final equality is given by the following bijection: fix $\sigma^{(i)} \in \Omega^{(i)}$, we have a bijection between the sets $C_{\mathfrak{t}_0} \cap d\mathfrak{S}_{\alpha|\beta}d^{-1}$ and $\Omega^{(i)}$ given by $\sigma' \mapsto \sigma^{(i)}\sigma'$. □

We are now ready to define, for each $\mathfrak{d} \in \mathscr{R}$, a map $\vartheta_{\mathfrak{d}}$ which will eventually lead to a generalisation of James' homomorphisms (Theorem 4.2).

Definition 4.11. Let λ be a partition of n. Recall that there is a natural left $\mathfrak{S}_n$-action on the set $\mathscr{T}(\lambda)$ of λ-tableaux, and let $\mathbb{Z}\mathscr{T}(\lambda)$ denote the associated permutation $\mathbb{Z}\mathfrak{S}_n$-module. Let $(\alpha|\beta)$ be a bicomposition of n and let Γ be a fixed left transversal of $\mathfrak{S}_{\alpha|\beta}$ in $\mathfrak{S}_n$. For each $\mathfrak{d} \in \mathscr{R}$, we define a $\mathbb{Z}$-linear map $\vartheta_{\mathfrak{d}} : \mathbb{Z}\mathscr{T}(\lambda) \to M_{\mathbb{Z}}(\alpha|\beta)$ as follows:

$$\vartheta_{\mathfrak{d}}(\mathfrak{t}) = \sum_{d\in\Gamma} \mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}}(d \otimes \mathbb{1} \otimes \epsilon),$$

where $\rho_{\mathfrak{t}} = \mathfrak{t} \circ (\mathfrak{t}_0)^{-1} \in \mathfrak{S}_n$ (so that $\rho_{\mathfrak{t}} \cdot \mathfrak{t}_0 = \rho_{\mathfrak{t}} \circ \mathfrak{t}_0 = \mathfrak{t}$).

The following properties of the map $\vartheta_{\mathfrak{d}}$ follow easily from its definition and Lemma 4.10(i).

Lemma 4.12. *Let* $\mathfrak{d} \in \mathscr{R}$.

(i) *The map* $\vartheta_{\mathfrak{d}}$ *is independent of the choice of the left transversal* Γ, *i.e. if* Γ' *is any left transversal of* $\mathfrak{S}_{\alpha|\beta}$ *in* $\mathfrak{S}_n$, *then*

$$\vartheta_{\mathfrak{d}}(\mathfrak{t}) = \sum_{d'\in\Gamma'} \mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d',\mathfrak{d}}(d' \otimes \mathbb{1} \otimes \epsilon)$$

for all $\mathfrak{t} \in \mathscr{T}(\lambda)$.

(ii) *If* $\mathfrak{d}' \in R_{\mathfrak{t}_0}\mathfrak{d}\mathfrak{S}_{\alpha|\beta}$, *then* $\vartheta_{\mathfrak{d}'} = \varepsilon_{\mathfrak{d}}(\mathfrak{d}')\vartheta_{\mathfrak{d}}$.

Proof. Let $d' = d\eta_\alpha\eta_\beta^{+|\alpha|}$ with $(\eta_\alpha, \eta_\beta) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$. Then $\mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d',\mathfrak{d}} = \mathrm{sgn}(\eta_\beta)\mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}}$ by Lemma 4.10(i) while

$$d' \otimes \mathbb{1} \otimes \epsilon = (d\eta_\alpha\eta_\beta) \otimes \mathbb{1} \otimes \epsilon = d \otimes (\eta_\alpha \cdot \mathbb{1}) \otimes (\eta_\beta \cdot \epsilon) = \mathrm{sgn}(\eta_\beta)(d \otimes \mathbb{1} \otimes \epsilon).$$

Thus $\mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d',\mathfrak{d}}(d' \otimes \mathbb{1} \otimes \epsilon) = \mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}}(d \otimes \mathbb{1} \otimes \epsilon)$ and part (i) follows.

By Lemma 4.10(i), $\mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}'} = \varepsilon_{\mathfrak{d}}(\mathfrak{d}')\mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}}$. Part (ii) thus follows. □

Next, we aim to show that each of the map $\vartheta_{\mathfrak{d}} : \mathbb{Z}\mathscr{T}(\lambda) \to M_{\mathbb{Z}}(\alpha|\beta)$ induces a $\mathbb{Z}\mathfrak{S}_n$-module homomorphism $\bar{\vartheta}_{\mathfrak{d}} : S^\lambda_{\mathbb{Z}} \to M_{\mathbb{Z}}(\alpha|\beta)$. For this, we will show that $\vartheta_{\mathfrak{d}}$ is a $\mathbb{Z}\mathfrak{S}_n$-module homomorphism and that the kernel of the natural map $\psi : \mathbb{Z}\mathscr{T}(\lambda) \to S^\lambda_{\mathbb{Z}}$ given by $\psi(\mathfrak{t}) = e_{\mathfrak{t}}$ is contained in $\ker(\vartheta_{\mathfrak{d}})$ and hence $\vartheta_{\mathfrak{d}}$ induces $\bar{\vartheta}_{\mathfrak{d}}$ as desired.

For each j, let $C_j(\lambda) = \{(i,j) \in [\lambda] : 1 \leq i \leq \ell(\lambda)\}$ be the j-th column of the Young diagram of λ. A *Garnir transversal* Δ is a left transversal of $\mathfrak{S}_X\mathfrak{S}_Y$ in $\mathfrak{S}_{X\cup Y}$, where $\emptyset \neq X \subseteq C_j(\lambda)$ and $\emptyset \neq Y \subseteq C_{j'}(\lambda)$ with $j < j'$, such that $|X|+|Y| > |C_j(\lambda)|$. Given a Garnir transversal Δ and a λ-tableau $\mathfrak{t}$, let $\Delta_{\mathfrak{t}} = \{\mathfrak{t} \circ \gamma \circ \mathfrak{t}^{-1} : \gamma \in \Delta\}$, so that $\Delta_{\mathfrak{t}}$ is a left transversal of $\mathfrak{S}_{\mathfrak{t}(X)}\mathfrak{S}_{\mathfrak{t}(Y)}$ in $\mathfrak{S}_{\mathfrak{t}(X\cup Y)}$, and write

$$G^{\mathfrak{t}}_\Delta = \sum_{\gamma\in\Delta_{\mathfrak{t}}} \mathrm{sgn}(\gamma)\gamma.$$

Proposition 4.13. *Let* $\mathfrak{d} \in \mathscr{R}$. *Then*

(i) $\vartheta_{\mathfrak{d}}$ *is a* $\mathbb{Z}\mathfrak{S}_n$*-module homomorphism, and*

(ii) *for any* $\mathfrak{t} \in \mathscr{T}(\lambda)$, $\pi \in C_{\mathfrak{t}}$ *and Garnir transversal* Δ, *we have that* $\ker(\vartheta_{\mathfrak{d}})$ *contains both* $\pi \cdot \mathfrak{t} - \mathrm{sgn}(\pi)\mathfrak{t}$ *and* $G^{\mathfrak{t}}_{\Delta} \cdot \mathfrak{t}$.

Proof. Let $\mathfrak{t} \in \mathscr{T}(\lambda)$ and $x \in \mathfrak{S}_n$. We have

$$\vartheta_{\mathfrak{d}}(x \cdot \mathfrak{t}) = \vartheta_{\mathfrak{d}}((x\rho_{\mathfrak{t}}) \cdot \mathfrak{t}_0) = \sum_{d \in \Gamma} \mathfrak{a}_{(x\rho_{\mathfrak{t}})^{-1}d,\mathfrak{d}}(d \otimes \mathbb{1} \otimes \epsilon),$$

$$x \cdot \vartheta_{\mathfrak{d}}(\mathfrak{t}) = x \cdot \left(\sum_{d \in \Gamma} \mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}}(d \otimes \mathbb{1} \otimes \epsilon) \right) = \sum_{d \in \Gamma} \mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}} \mathrm{sgn}(\xi_{d,\beta})(f(d) \otimes \mathbb{1} \otimes \epsilon),$$

where $xd = f(d)\xi_{d,\alpha}\xi_{d,\beta}^{+|\alpha|}$ with $f(d) \in \Gamma$, $(\xi_{d,\alpha}, \xi_{d,\beta}) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$. As such, we need to show that for each $d \in \Gamma$,

$$\mathfrak{a}_{(x\rho_{\mathfrak{t}})^{-1}f(d),\mathfrak{d}} = \mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}} \mathrm{sgn}(\xi_{d,\beta}).$$

Observe that $\sigma(x\rho_{\mathfrak{t}})^{-1}f(d) = \tau\mathfrak{d}\eta_\alpha\eta_\beta^{+|\alpha|}$ for some $\sigma \in C_{\mathfrak{t}_0}$, $\tau \in R_{\mathfrak{t}_0}$ and $(\eta_\alpha, \eta_\beta) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$ if and only if $\sigma\rho_{\mathfrak{t}}^{-1}d = \tau\mathfrak{d}(\eta_\alpha\xi_{d,\alpha})(\eta_\beta\xi_{d,\beta})^{+|\alpha|}$ for some $\sigma \in C_{\mathfrak{t}_0}$, $\tau \in R_{\mathfrak{t}_0}$ and $(\eta_\alpha, \eta_\beta) \in \mathfrak{S}_\alpha \times \mathfrak{S}_\beta$, and in which case, $\varepsilon_{\mathfrak{d}}(\sigma(\rho_{\mathfrak{t}})^{-1}d) = \mathrm{sgn}(\xi_{d,\beta})\varepsilon_{\mathfrak{d}}(\sigma(x\rho_{\mathfrak{t}})^{-1}f(d))$. So $\Omega_{(x\rho_{\mathfrak{t}})^{-1}f(d),\mathfrak{d}} = \Omega_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}}$, and

$$\begin{aligned}
\mathfrak{a}_{(x\rho_{\mathfrak{t}})^{-1}f(d),\mathfrak{d}} &= \sum_{\sigma \in \Omega_{(x\rho_{\mathfrak{t}})^{-1}f(d),\mathfrak{d}}} \mathrm{sgn}(\sigma)\varepsilon_{\mathfrak{d}}(\sigma(x\rho_{\mathfrak{t}})^{-1}f(d)) \\
&= \sum_{\sigma \in \Omega_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}}} \mathrm{sgn}(\sigma)\mathrm{sgn}(\xi_{d,\beta})\varepsilon_{\mathfrak{d}}(\sigma\rho_{\mathfrak{t}}^{-1}d) \\
&= \mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}} \mathrm{sgn}(\xi_{d,\beta}).
\end{aligned}$$

This proves part (i).

Since $\rho_{\mathfrak{t}} \cdot \mathfrak{t}_0 = \mathfrak{t}$, we have $C_{\mathfrak{t}} = \rho_{\mathfrak{t}} C_{\mathfrak{t}_0} \rho_{\mathfrak{t}}^{-1}$ and hence $\rho_{\mathfrak{t}}^{-1}\pi^{-1}\rho_{\mathfrak{t}} \in C_{\mathfrak{t}_0}$. By Lemma 4.10(i), we have

$$\mathfrak{a}_{(\pi\rho_{\mathfrak{t}})^{-1}d,\mathfrak{d}} = \mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}\pi^{-1}\rho_{\mathfrak{t}}\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}} = \mathrm{sgn}(\pi)\mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}}.$$

Therefore

$$\begin{aligned}
\vartheta_{\mathfrak{d}}(\pi \cdot \mathfrak{t}) &= \sum_{d \in \Gamma} \mathfrak{a}_{(\pi\rho_{\mathfrak{t}})^{-1}d,\mathfrak{d}}(d \otimes \mathbb{1} \otimes \epsilon) \\
&= \sum_{d \in \Gamma} \mathrm{sgn}(\pi)\mathfrak{a}_{\rho_{\mathfrak{t}}^{-1}d,\mathfrak{d}}(d \otimes \mathbb{1} \otimes \epsilon) \\
&= \mathrm{sgn}(\pi)\vartheta_{\mathfrak{d}}(\mathfrak{t}).
\end{aligned}$$

Next we turn to $G^{\mathfrak{t}}_{\Delta}\cdot\mathfrak{t}$. We have

$$\begin{aligned}
&\vartheta_{\mathfrak{d}}(G^{\mathfrak{t}}_{\Delta}\cdot\mathfrak{t})\\
&=\vartheta_{\mathfrak{d}}\left(\sum_{\gamma\in\Delta_{\mathfrak{t}}}\mathrm{sgn}(\gamma)(\gamma\cdot\mathfrak{t})\right)\\
&=\sum_{\gamma\in\Delta_{\mathfrak{t}}}\sum_{d\in\Gamma}\mathrm{sgn}(\gamma)\mathfrak{a}_{(\gamma\rho_{\mathfrak{t}})^{-1}d,\mathfrak{d}}(d\otimes\mathbb{1}\otimes\epsilon)\\
&=\sum_{d\in\Gamma}\left(\sum_{\gamma\in\Delta_{\mathfrak{t}}}\sum_{\sigma\in\Omega_{(\gamma\rho_{\mathfrak{t}})^{-1}d,\mathfrak{d}}}\mathrm{sgn}(\gamma)\mathrm{sgn}(\sigma)\varepsilon_{\mathfrak{d}}(\sigma(\gamma\rho_{\mathfrak{t}})^{-1}d)\right)(d\otimes\mathbb{1}\otimes\epsilon).
\end{aligned}$$

Fix $d\in\Gamma$, and let $b(\gamma,\sigma)=\mathrm{sgn}(\gamma)\mathrm{sgn}(\sigma)\varepsilon_{\mathfrak{d}}(\sigma(\gamma\rho_{\mathfrak{t}})^{-1}d)$. We need to show that $\sum_{(\gamma,\sigma)\in\Upsilon}b(\gamma,\sigma)=0$, where $\Upsilon=\{(\gamma,\sigma):\gamma\in\Delta_{\mathfrak{t}},\ \sigma\in\Omega_{(\gamma\rho_{\mathfrak{t}})^{-1}d,\mathfrak{d}}\}$.

Let the Garnir transversal Δ be a left transversal of $\mathfrak{S}_X\mathfrak{S}_Y$ in $\mathfrak{S}_{X\cup Y}$, where X and Y are subsets of the jth and j'th column of $[\lambda]$, with $|X|+|Y|>|C_j(\lambda)|$. Since $|X|+|Y|>|C_j(\lambda)|$, there exists $1\le i\le\ell(\lambda)$ such that $(i,j)\in X$ and $(i,j')\in Y$, and we may choose i to be the least such. Let η be the transposition $(\mathfrak{t}_0(i,j)\ \ \mathfrak{t}_0(i,j'))$. Then $\eta\in R_{\mathfrak{t}_0}$.

Let $(\gamma,\sigma)\in\Upsilon$, say $\sigma(\gamma\rho_{\mathfrak{t}})^{-1}d=\tau\mathfrak{d}\xi_\alpha\xi_\beta^{+|\alpha|}$ where $\tau\in R_{\mathfrak{t}_0}$ and $(\xi_\alpha,\xi_\beta)\in\mathfrak{S}_\alpha\times\mathfrak{S}_\beta$. Then

$$\begin{aligned}
R_{\mathfrak{t}_0}\mathfrak{d}\mathfrak{S}_{\alpha|\beta}\ni\eta\tau\mathfrak{d}\xi_\alpha\xi_\beta^{+|\alpha|}&=\eta\sigma(\gamma\rho_{\mathfrak{t}})^{-1}d=\eta\sigma\rho_{\mathfrak{t}}^{-1}\gamma^{-1}d\\
&=(\sigma\rho_{\mathfrak{t}}^{-1})((\sigma\rho_{\mathfrak{t}}^{-1})^{-1}\eta(\sigma\rho_{\mathfrak{t}}^{-1}))\gamma^{-1}d\\
&=\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1}\rho_{\mathfrak{t}}(\gamma'\rho_{\mathfrak{t}})^{-1}d,
\end{aligned}$$

where $\gamma'\in\Delta_{\mathfrak{t}}$ and $\zeta\in\mathfrak{S}_{\mathfrak{t}(X)}\mathfrak{S}_{\mathfrak{t}(Y)}$ satisfies $\gamma((\sigma\rho_{\mathfrak{t}}^{-1})^{-1}\eta(\sigma\rho_{\mathfrak{t}}^{-1}))=\gamma'\zeta$. Then $\rho_{\mathfrak{t}}^{-1}\zeta^{-1}\rho_{\mathfrak{t}}\in\rho_{\mathfrak{t}}^{-1}\mathfrak{S}_{\mathfrak{t}(X)}\mathfrak{S}_{\mathfrak{t}(Y)}\rho_{\mathfrak{t}}=\mathfrak{S}_{\mathfrak{t}_0(X)}\mathfrak{S}_{\mathfrak{t}_0(Y)}\subseteq C_{\mathfrak{t}_0}$. This shows that $(\gamma',\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1}\rho_{\mathfrak{t}})\in\Upsilon$. In other words, the function $h:\Upsilon\to\Upsilon$ defined by $(\gamma,\sigma)\mapsto(\gamma',\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1}\rho_{\mathfrak{t}})$, where $\gamma((\sigma\rho_{\mathfrak{t}}^{-1})^{-1}\eta(\sigma\rho_{\mathfrak{t}}^{-1}))=\gamma'\zeta$, is well-defined. Furthermore,

$$\begin{aligned}
b(h(\gamma,\sigma))&=\mathrm{sgn}(\gamma')\mathrm{sgn}(\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1}\rho_{\mathfrak{t}})\varepsilon_{\mathfrak{d}}(\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1}\rho_{\mathfrak{t}}(\gamma'\rho_{\mathfrak{t}})^{-1}d)\\
&=-\mathrm{sgn}(\gamma)\mathrm{sgn}(\sigma)\mathrm{sgn}(\xi_\beta)\\
&=-b(\gamma,\sigma).
\end{aligned}$$

We claim that h is a fixed-point-free involution, in which case, since the contributions from (γ,σ) and $h(\gamma,\sigma)$ towards the sum $\sum_{(\gamma,\sigma)\in\Upsilon}b(\gamma,\sigma)$ cancel each other out, we have $\sum_{(\gamma,\sigma)\in\Upsilon}b(\gamma,\sigma)=0$ as desired.

To prove the claim, first suppose that

$$(\gamma,\sigma)=h(\gamma,\sigma)=(\gamma',\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1}\rho_{\mathfrak{t}}).$$

Then $\gamma=\gamma'$ and $\zeta=1$ and hence $\eta=1$, a contradiction. So h is fixed-point-free. Next, $h^2(\gamma,\sigma)=h(\gamma',\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1}\rho_{\mathfrak{t}})=(\gamma'',\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1}\rho_{\mathfrak{t}}\rho_{\mathfrak{t}}^{-1}\zeta'^{-1}\rho_{\mathfrak{t}})$, where

$$\gamma''\zeta'=\gamma'(\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1})^{-1}\eta(\sigma\rho_{\mathfrak{t}}^{-1}\zeta^{-1})=\gamma'\zeta(\sigma\rho_{\mathfrak{t}}^{-1})^{-1}\eta(\sigma\rho_{\mathfrak{t}}^{-1})\zeta^{-1}=\gamma\zeta^{-1}.$$

Thus $\gamma'' = \gamma$ and $\zeta' = \zeta^{-1}$ and hence $h^2(\gamma,\sigma) = (\gamma,\sigma)$, and the proof is complete. $\square$

The next result is well known when the underlying ring is a field (see, for example, [F, §7.4, Corollary, pg 101]); we are however unable to find its generalisation to $\mathbb{Z}$ in the existing literature.

Lemma 4.14. *The map* $\psi : \mathbb{Z}\mathscr{T}(\lambda) \to S^\lambda_\mathbb{Z}$ *defined by* $\mathfrak{t} \mapsto e_\mathfrak{t}$ *is a* $\mathbb{Z}\mathfrak{S}_n$*-module epimorphism, and* $\ker(\psi)$ *is generated, as a* $\mathbb{Z}$*-submodule of* $\mathbb{Z}\mathscr{T}(\lambda)$*, by* $G \cup H$*, where*

$$G = \{G^\mathfrak{t}_\Delta \cdot \mathfrak{t} : \mathfrak{t} \in \mathscr{T}(\lambda),\ \Delta \text{ a Garnir transversal}\},$$
$$H = \{\pi \cdot \mathfrak{t} - \operatorname{sgn}(\pi)\mathfrak{t} : \mathfrak{t} \in \mathscr{T}(\lambda),\ \pi \in C_\mathfrak{t}\}.$$

Proof. That ψ is a $\mathbb{Z}\mathfrak{S}_n$-module epimorphism is clear, so we only need to justify the assertion about its kernel. Let $K = \mathbb{Z}(G \cup H)$. It is straightforward to verify that both G and H are invariant under the action of $\mathfrak{S}_n$, and that $\psi(G \cup H) = \{0\}$ [JK, 7.2.1, 7.2.3], so that K is a $\mathbb{Z}\mathfrak{S}_n$-submodule of $\mathbb{Z}\mathscr{T}(\lambda)$, and $K \subseteq \ker(\psi)$. It remains to show that $\ker(\psi) \subseteq K$.

Since $\psi(\mathbb{Z}\mathscr{T}(\lambda)) = S^\lambda_\mathbb{Z}$, and $S^\lambda_\mathbb{Z}$ is $\mathbb{Z}$-free with basis $\{e_\mathfrak{t} : \mathfrak{t} \in \mathscr{T}_{\mathrm{std}}(\lambda)\}$ [P2, Theorem 1.1], we see that $\mathbb{Z}\mathscr{T}(\lambda) = (\bigoplus_{\mathfrak{t}\in\mathscr{T}_{\mathrm{std}}(\lambda)} \mathbb{Z}\mathfrak{t}) \oplus \ker(\psi)$.

Consider $\mathbb{Z}\mathscr{T}(\lambda)/K$. For each $\mathfrak{t} \in \mathscr{T}(\lambda)$, let $v_\mathfrak{t} = \mathfrak{t} + K \in \mathbb{Z}\mathscr{T}(\lambda)/K$. Then $\pi \cdot v_\mathfrak{t} = \operatorname{sgn}(\pi)v_\mathfrak{t}$ for any $\pi \in C_\mathfrak{t}$, and $G^\mathfrak{t}_\Delta \cdot v_\mathfrak{t} = 0$ for any Garnir transversal Δ. Using the same argument as in the proof of [JK, 7.2.7], we can show that $\{v_\mathfrak{t} : \mathfrak{t} \in \mathscr{T}_{\mathrm{std}}(\lambda)\}$ generates $\mathbb{Z}\mathscr{T}(\lambda)/K$ as a $\mathbb{Z}$-module.

Let $x \in \ker(\psi)$. Then there exists $b_\mathfrak{t} \in \mathbb{Z}$ for each $\mathfrak{t} \in \mathscr{T}_{\mathrm{std}}(\lambda)$ such that

$$x + K = \sum_{\mathfrak{t}\in\mathscr{T}_{\mathrm{std}}(\lambda)} b_\mathfrak{t} v_\mathfrak{t} = \sum_{\mathfrak{t}\in\mathscr{T}_{\mathrm{std}}(\lambda)} b_\mathfrak{t}(\mathfrak{t} + K).$$

Thus there exists $k \in K$ such that

$$x + k = \sum_{\mathfrak{t}\in\mathscr{T}_{\mathrm{std}}(\lambda)} b_\mathfrak{t}\mathfrak{t} \in \left(\bigoplus_{\mathfrak{t}\in\mathscr{T}_{\mathrm{std}}(\lambda)} \mathbb{Z}\mathfrak{t}\right) \cap \ker(\psi) = \{0\}.$$

Hence $x + k = 0$ and so $x \in K$, and our proof is complete. $\square$

The following is the main theorem of this section.

Theorem 4.15. *Let* λ *be a partition of* n*,* $(\alpha|\beta)$ *be a bicomposition of* n *and* Γ *be a left transversal of* $\mathfrak{S}_{\alpha|\beta}$ *in* $\mathfrak{S}_n$*. For each* $\mathfrak{d} \in \mathscr{R}$*, we have a* $\mathbb{Z}\mathfrak{S}_n$*-module homomorphism* $\bar{\vartheta}_\mathfrak{d} : S^\lambda_\mathbb{Z} \to M_\mathbb{Z}(\alpha|\beta)$ *given by*

$$\bar{\vartheta}_\mathfrak{d}(e_\mathfrak{t}) = \sum_{d\in\Gamma} \mathfrak{a}_{\rho_\mathfrak{t}^{-1}d,\mathfrak{d}}\,(d \otimes \mathbb{1} \otimes \epsilon).$$

Proof. By Proposition 4.13 and Lemma 4.14, we have $\ker(\psi) \subseteq \ker(\vartheta_\mathfrak{d})$, so that there is a unique $\mathbb{Z}\mathfrak{S}_n$-module homomorphism $\bar{\vartheta}_\mathfrak{d} : S^\lambda_\mathbb{Z} \to M_\mathbb{Z}(\alpha|\beta)$ such that $\bar{\vartheta}_\mathfrak{d} \circ \psi = \vartheta_\mathfrak{d}$. The theorem immediately follows. $\square$

Let $\mathbb{F}$ be a field. Since

$$S^\lambda_{\mathbb{F}} = \mathbb{F} \otimes_{\mathbb{Z}} S^\lambda_{\mathbb{Z}} \qquad \text{and} \qquad M_{\mathbb{F}}(\alpha|\beta) \cong \mathbb{F} \otimes_{\mathbb{Z}} M_{\mathbb{Z}}(\alpha|\beta),$$

each $\bar{\vartheta}_{\mathscr{d}}$ gives rise to an $\mathbb{F}\mathfrak{S}_n$-homomorphism $\bar{\vartheta}^{\mathbb{F}}_{\mathscr{d}} : S^\lambda_{\mathbb{F}} \to M_{\mathbb{F}}(\alpha|\beta)$. More specifically, for any $k \in \mathbb{Z}$, let $k^{\mathbb{F}} = k \cdot 1_{\mathbb{F}} \in \mathbb{F}$. Then

$$\bar{\vartheta}^{\mathbb{F}}_{\mathscr{d}}(e_{\mathfrak{t}}) = \sum_{d \in \Gamma} \mathfrak{a}^{\mathbb{F}}_{\rho_{\mathfrak{t}}^{-1} d, \mathscr{d}} (d \otimes \mathbb{1} \otimes \epsilon).$$

It is not difficult to see that our $\bar{\vartheta}^{\mathbb{F}}_{\mathscr{d}}$'s generalise the $\hat{\theta}_{\mathrm{T}}$'s constructed by James. Indeed, by comparing Theorems 4.2 and 4.15, we see that when $(\alpha|\beta) = (\mu|\varnothing)$, we have $\mathscr{R} = \mathfrak{S}_n$ and, for all $\mathrm{T} \in \mathscr{T}(\lambda,\mu) = \mathscr{T}(\lambda,(\mu|\varnothing))$, $\mathfrak{t} \in \mathscr{T}(\lambda)$ and $d \in \Gamma$, $a_{\rho_{\mathfrak{t}}^{-1} d, d_{\mathrm{T}}} = \mathfrak{a}^{\mathbb{F}}_{\rho_{\mathfrak{t}}^{-1} d, d_{\mathrm{T}}}$ and $\hat{\theta}_{\mathrm{T}} = \bar{\vartheta}^{\mathbb{F}}_{d_{\mathrm{T}}}$ (recall that $\mathrm{T} = d_{\mathrm{T}} \cdot \mathrm{T}_0$).

5. Basis

Let λ be a partition of n and let $(\alpha|\beta)$ be a bicomposition of n. As before, fix a λ-tableau $\mathfrak{t}_0$ so that $\mathfrak{S}_n$ acts on $\mathscr{T}(\lambda,(\alpha|\beta))$ through $\mathfrak{t}_0$. Fix a left transversal Γ of $\mathfrak{S}_{\alpha|\beta}$ in $\mathfrak{S}_n$, and write

$$\Gamma_{\mathrm{sstd}} = \{d \in \Gamma : \mathrm{T}_d \in \mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))\}.$$

Recall from Corollary 4.6 that we have $\Gamma_{\mathrm{sstd}} \subseteq \mathscr{R} \cap \mathscr{C}$. It is easy to see that $|\Gamma_{\mathrm{sstd}}| = |\mathscr{T}_{\mathrm{sstd}}(\lambda,(\alpha|\beta))|$.

In the last section, we constructed for each $\mathscr{d} \in \mathscr{R}$ an $\mathbb{F}\mathfrak{S}_n$-homomorphism $\bar{\vartheta}^{\mathbb{F}}_{\mathscr{d}} \in \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$ which generalises James's construction of $\hat{\theta}_{\mathrm{T}} \in \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M^\mu_{\mathbb{F}})$ where $\mathrm{T} \in \mathscr{T}(\lambda,\mu)$. While James's θ_{T} is non-zero in any characteristic, our $\bar{\vartheta}^{\mathbb{F}}_{\mathscr{d}}$ ($\mathscr{d} \in \mathscr{R}$), in view of Lemma 4.10(iii), may be zero in some characteristic. Let

$$g_{\mathscr{d}} := \gcd_{\mathbb{Z}}\{\mathfrak{a}_{d,\mathscr{d}} : d \in \Gamma\} \qquad \text{and} \qquad \hat{\vartheta}_{\mathscr{d}} := \tfrac{1}{g_{\mathscr{d}}} \bar{\vartheta}_{\mathscr{d}}.$$

Observe that $g_{\mathscr{d}}$ is independent of the left transversal Γ by Lemma 4.10(i). Furthermore, $\hat{\vartheta}_{\mathscr{d}} \in \mathrm{Hom}_{\mathbb{Z}\mathfrak{S}_n}(S^\lambda_{\mathbb{Z}}, M_{\mathbb{Z}}(\alpha|\beta))$, and

$$\hat{\vartheta}_{\mathscr{d}}(e_{\mathfrak{t}_0}) = \sum_{d \in \Gamma} \hat{\mathfrak{a}}_{d,\mathscr{d}} (d \otimes \mathbb{1} \otimes \epsilon),$$

where $\hat{\mathfrak{a}}_{d,\mathscr{d}} := \frac{\mathfrak{a}_{d,\mathscr{d}}}{g_{\mathscr{d}}} \in \mathbb{Z}$ for all $d \in \Gamma$. From $\hat{\vartheta}_{\mathscr{d}}$, we obtain $\hat{\vartheta}^{\mathbb{F}}_{\mathscr{d}} \in \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$, where

$$\hat{\vartheta}^{\mathbb{F}}_{\mathscr{d}}(e_{\mathfrak{t}_0}) = \sum_{d \in \Gamma} \hat{\mathfrak{a}}^{\mathbb{F}}_{d,\mathscr{d}} (d \otimes \mathbb{1} \otimes \epsilon).$$

Clearly, $\hat{\vartheta}^{\mathbb{F}}_{\mathscr{d}} \neq 0$ by construction, irrespective of the characteristic of $\mathbb{F}$, and $\bar{\vartheta}^{\mathbb{F}}_{\mathscr{d}} = g^{\mathbb{F}}_{\mathscr{d}} \hat{\vartheta}^{\mathbb{F}}_{\mathscr{d}}$ where $g^{\mathbb{F}}_{\mathscr{d}} = 1_{\mathbb{F}} \cdot g_{\mathscr{d}}$. We introduce the following notations.

Notation 5.1. Let λ be a partition and $(\alpha|\beta)$ be a bicomposition of n. Fix a λ-tableau $\mathfrak{t}_0$. We denote

$$\overline{\Theta}^{\mathbb{F}}_{\mathscr{R}} := \{\bar{\vartheta}^{\mathbb{F}}_{\mathscr{d}} : \mathscr{d} \in \mathscr{R}\}, \qquad \overline{\Theta}^{\mathbb{F}}_{\mathrm{sstd}} := \{\bar{\vartheta}^{\mathbb{F}}_{\mathscr{d}} \in \overline{\Theta}^{\mathbb{F}}_{\mathscr{R}} : \mathscr{d} \in \Gamma_{\mathrm{sstd}}\},$$
$$\widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}} := \{\hat{\vartheta}^{\mathbb{F}}_{\mathscr{d}} : \mathscr{d} \in \mathscr{R}\}, \qquad \widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}} := \{\hat{\vartheta}^{\mathbb{F}}_{\mathscr{d}} \in \widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}} : \mathscr{d} \in \Gamma_{\mathrm{sstd}}\},$$

where $\hat{\vartheta}_{\mathfrak{d}} = \frac{1}{g_{\mathfrak{d}}}\bar{\vartheta}_{\mathfrak{d}}$, $\hat{\vartheta}^{\mathbb{F}}_{\mathfrak{d}} = 1_{\mathbb{F}} \cdot \hat{\vartheta}_{\mathfrak{d}}$ and $g_{\mathfrak{d}} = \gcd_{\mathbb{Z}}\{\mathfrak{a}_{d,\mathfrak{d}} : d \in \Gamma\}$.

In James's classical case, i.e. when $\beta = \varnothing$, we have $\overline{\Theta}^{\mathbb{F}}_{\mathscr{R}} = \overline{\Theta}^{\mathbb{F}}_{\mathfrak{S}_n} = \widehat{\Theta}^{\mathbb{F}}_{\mathfrak{S}_n}$ and

$$\overline{\Theta}^{\mathbb{F}}_{\text{sstd}} = \widehat{\Theta}^{\mathbb{F}}_{\text{sstd}} = \{\hat{\theta}_{\mathrm{T}} : \mathrm{T} \in \mathscr{T}_{\text{sstd}}(\lambda, \alpha)\}.$$

Furthermore, he showed that $\overline{\Theta}^{\mathbb{F}}_{\text{sstd}}$ is a basis for $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^{\lambda}_{\mathbb{F}}, M^{\alpha}_{\mathbb{F}})$, unless $\mathbb{F}$ has characteristic 2 and λ is 2-singular (see [J, 13.13 Theorem]). This is therefore a characteristic-free basis of $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^{\lambda}_{\mathbb{F}}, M^{\alpha}_{\mathbb{F}})$, as long as the characteristic is not 2.

One may hope that our generalisation $\overline{\Theta}^{\mathbb{F}}_{\text{sstd}}$ will also provide a characteristic-free basis for $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^{\lambda}_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$. However, such hopes are misplaced, as the dimension of $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^{\lambda}_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$ is not characteristic-free in general. Indeed, in [J, 24.4 Theorem], a necessary and sufficient condition in terms of the characteristic of $\mathbb{F}$ (as well as the partition λ) is obtained for $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, S^{\lambda}_{\mathbb{F}})$ to be non-zero. Since

$$\begin{aligned}\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathbb{F}, S^{\lambda'}_{\mathbb{F}}) &\cong \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathrm{sgn}, S^{\lambda'}_{\mathbb{F}} \otimes \mathrm{sgn}) \\ &\cong \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(\mathrm{sgn}, (S^{\lambda}_{\mathbb{F}})^*) \\ &\cong \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^{\lambda}_{\mathbb{F}}, \mathrm{sgn}) = \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^{\lambda}_{\mathbb{F}}, M_{\mathbb{F}}(\varnothing|(n))),\end{aligned}$$

we see that, when $(\alpha|\beta) = (\varnothing|(n))$, $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^{\lambda}_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$ is zero or not is dependent of the characteristic of $\mathbb{F}$. We give another example below.

Example 5.2. Let ℓ be an odd prime integer, and let $\mathbb{F}$ have characteristic p, where p is either odd or zero. Let $\lambda = (\ell, 2, 1^{\ell-2})$ and $\alpha = (\ell) = \beta$. Clearly, $\mathscr{T}_{\text{sstd}}((\ell, 2, 1^{\ell-2}), ((\ell)|(\ell))) = \emptyset$.

Firstly, $M_{\mathbb{F}}(\alpha|\beta) = \mathrm{Ind}^{\mathfrak{S}_{2\ell}}_{\mathfrak{S}_\ell \times \mathfrak{S}_\ell}(\mathbb{F} \boxtimes \mathrm{sgn}) \cong \mathrm{Ind}^{\mathfrak{S}_{2\ell}}_{\mathfrak{S}_\ell \times \mathfrak{S}_\ell}(S^{(\ell)}_{\mathbb{F}} \boxtimes S^{(1^\ell)}_{\mathbb{F}})$ has a Specht filtration with two factors: $S^{(\ell+1,1^{\ell-1})}_{\mathbb{F}}$ at the top and $S^{(\ell,1^\ell)}_{\mathbb{F}}$ at the bottom [JP], which are simple unless $p = \ell$ [P1, Theorem 2]. Furthermore, unless $p = \ell$, the partitions $(\ell, 2, 1^{\ell-2})$, $(\ell+1, 1^{\ell-1})$ and $(\ell, 1^\ell)$ have distinct p-cores and hence the Specht modules they label lie in distinct blocks of $\mathfrak{S}_{2\ell}$ by the Nakayama Rule [JK, 6.1.42 Corollary], so that $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_{2\ell}}(S^{\lambda}_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta)) = 0$.

On the other hand, if $p = \ell$, then $S^{(\ell,1^\ell)}_{\mathbb{F}}$, a submodule of $M_{\mathbb{F}}(\alpha|\beta)$, is a non-split extension of $D^{(\ell,2,1^{\ell-2})}$ by $D^{(\ell+1,1^{\ell-1})}$. From the known radical structures of Specht modules lying in defect 2 blocks of symmetric group algebras (see [R, Theorem 4.4] and [CT, Proposition 6.2]), we see that $S^{\lambda}_{\mathbb{F}}$ has a quotient which is a non-split extension of $D^{(\ell,2,1^{\ell-2})}$ by $D^{(\ell+1,1^{\ell-1})}$ as well. Since all nonzero extensions between simple modules lying in defect 2 blocks of symmetric group algebras are one-dimensional [S, Theorem I(5)], we see that $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_{2\ell}}(S^{\lambda}_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta)) \neq 0$.

Thus, $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_{2\ell}}(S^{\lambda}_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta)) \neq 0$ if and only if $p = \ell$.

Our best hope is for $\widehat{\Theta}^{\mathbb{F}}_{\text{sstd}}$ to be a basis for the $\mathbb{F}$-span of $\widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}}$. However, the next example shows that this simplistic attempt is insufficient in ensuring that the set $\widehat{\Theta}^{\mathbb{F}}_{\text{sstd}}$ is linearly independent.

Example 5.3. Suppose that $\mathbb{F}$ have positive characteristic $p \geq 3$, and let $\lambda = (2, 1^{p+2})$ and $(\alpha|\beta) = (\varnothing|(p, 2^2))$. Also, let $\mathfrak{t}_0$ be such that $\mathfrak{t}_0(j, 1) = j$ for $1 \leq j \leq p+3$ and $\mathfrak{t}_0(1, 2) = p+4$. Thus,

$$\mathrm{T}_0(i,j) = \begin{cases} \mathbb{d}_1, & \text{if } 1 \leq i \leq p \text{ and } j = 1; \\ \mathbb{d}_2, & \text{if } p+1 \leq i \leq p+2 \text{ and } j = 1; \\ \mathbb{d}_3, & \text{if } (i,j) \in \{(p+3,1),(1,2)\}. \end{cases}$$

For each $i \in \{1, 2, 3\}$, let

$$\mathcal{T}_i = \{\mathrm{T} \in \mathscr{T}(\lambda, (\alpha|\beta)) : \mathrm{T}(1,2) = \mathbb{d}_i\}.$$

Then the $\mathcal{T}_i$'s are the orbits of $\mathscr{T}(\lambda, (\alpha|\beta))$ under the action of $C_{\mathfrak{t}_0}$.

There is no semistandard λ-tableau of type $(\alpha|\beta)$ in $\mathcal{T}_1$, and exactly one semistandard λ-tableau of type $(\alpha|\beta)$, denoted T_2 and T_3, in each of $\mathcal{T}_2$ and $\mathcal{T}_3$ respectively. Let $d_1 = (p\ \ p+4\ \ p+2)$, $d_2 = (p+2\ \ p+4)$ and $d_3 = 1_{\mathfrak{S}_{p+4}}$ be permutations in $\mathfrak{S}_{p+4}$. Let $\mathrm{T}_1 := d_1 \cdot \mathrm{T}_0 \in \mathcal{T}_1$. Then $d_i \cdot \mathrm{T}_0 = \mathrm{T}_i$ for $i \in \{1, 2, 3\}$. Extend $\{d_1, d_2, d_3\}$ to a left transversal Γ of $\mathfrak{S}_{\alpha|\beta}$ in $\mathfrak{S}_{p+4}$.

Using Lemma 4.10(iii), we have

$$\mathfrak{a}_{d_1,d_2} = (p-1)! \cdot 4, \qquad \mathfrak{a}_{d_2,d_2} = p! \cdot 2, \qquad \mathfrak{a}_{d_3,d_2} = 0.$$

Thus, by Lemma 4.10(i) and the remark above, for any $d \in \Gamma$, we have $\mathfrak{a}_{d,d_2} = \varepsilon \mathfrak{a}_{d_i,d_2}$ for some $i = 1, 2, 3$ and $\varepsilon = \pm 1$. Therefore, $g_{d_2} = \gcd_{\mathbb{Z}}\{\mathfrak{a}_{d,d_2} : d \in \Gamma\} = (p-1)! \cdot 2$. Thus, if $\hat{\vartheta}^{\mathbb{F}}_{d_2}(e_{\mathfrak{t}_0}) = \sum_{d \in \Gamma} b_d(d \otimes \mathbb{1} \otimes \epsilon)$, then $b_{d_1} = 2$ and $b_{d_2} = 0 = b_{d_3}$.

Similarly,

$$\mathfrak{a}_{d_1,d_3} = -(p-1)! \cdot 4, \qquad \mathfrak{a}_{d_2,d_3} = 0, \qquad \mathfrak{a}_{d_3,d_3} = p! \cdot 2,$$

so that if $\hat{\vartheta}^{\mathbb{F}}_{d_3}(e_{\mathfrak{t}_0}) = \sum_{d \in \Gamma} c_d(d \otimes \mathbb{1} \otimes \epsilon)$, then $c_{d_1} = -2$ and $c_{d_2} = 0 = c_{d_3}$.

If $d \in \Gamma$, say with $d \cdot \mathrm{T}_0 \in \mathcal{T}_i$, then $d \cdot \mathrm{T}_0 = \sigma \cdot \mathrm{T}_i = (\sigma d_i) \cdot \mathrm{T}_0$ for some $\sigma \in C_{\mathfrak{t}_0}$, so that $d = \sigma d_i \xi$ for some $\xi \in \mathfrak{S}_{\alpha|\beta} = \mathfrak{S}_\beta$. Thus, $\mathfrak{a}_{d,d_2} = \mathrm{sgn}(\sigma)\mathrm{sgn}(\xi)\mathfrak{a}_{d_i,d_2}$, and $\mathfrak{a}_{d,d_3} = \mathrm{sgn}(\sigma)\mathrm{sgn}(\xi)\mathfrak{a}_{d_i,d_3}$ by Lemma 4.10(i), so that $b_d = -c_d$. Consequently, $\hat{\vartheta}^{\mathbb{F}}_{d_2} + \hat{\vartheta}^{\mathbb{F}}_{d_3} = 0$, and $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}} = \{\hat{\vartheta}^{\mathbb{F}}_{d_2}, \hat{\vartheta}^{\mathbb{F}}_{d_3}\}$ is not linearly independent.

However, when $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ is linearly independent, it necessarily will be a basis for the $\mathbb{F}$-span of $\widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}}$, as the following result shows.

Proposition 5.4. *Let $|\mathscr{T}_{\mathrm{sstd}}(\lambda, (\alpha|\beta))| = k$. Let $d_1, d_2, \ldots, d_k \in \mathscr{R}$ and suppose that $\hat{\vartheta}^{\mathbb{F}}_{d_1}, \hat{\vartheta}^{\mathbb{F}}_{d_2}, \ldots, \hat{\vartheta}^{\mathbb{F}}_{d_k}$ are linearly independent. Then $\{\hat{\vartheta}^{\mathbb{F}}_{d_1}, \hat{\vartheta}^{\mathbb{F}}_{d_2}, \ldots, \hat{\vartheta}^{\mathbb{F}}_{d_k}\}$ is a basis for the $\mathbb{F}$-span of $\widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}}$.*

Furthermore, if $S^\lambda_{\mathbb{F}}$ lies in a block of $\mathbb{F}\mathfrak{S}_n$ which is simple as an algebra, then $\{\hat{\vartheta}^{\mathbb{F}}_{d_1}, \hat{\vartheta}^{\mathbb{F}}_{d_2}, \ldots, \hat{\vartheta}^{\mathbb{F}}_{d_k}\}$ is a basis of $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$.

Proof. We prove the second assertion first. If $S^\lambda_{\mathbb{F}}$ lies in a block of $\mathbb{F}\mathfrak{S}_n$ which is simple as an algebra, then the composition multiplicity of $S^\lambda_{\mathbb{F}}$ in $M_{\mathbb{F}}(\alpha|\beta)$ equals $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$. By Theorem 3.5, this must be equal to $|\mathscr{T}_{\mathrm{sstd}}(\lambda, (\alpha|\beta))| = k$. Since $\{\hat{\vartheta}^{\mathbb{F}}_{d_1}, \hat{\vartheta}^{\mathbb{F}}_{d_2}, \ldots, \hat{\vartheta}^{\mathbb{F}}_{d_k}\}$ is a linearly independent subset of $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$, it is a basis.

Now we turn our attention to the first assertion. Let $d_0 \in \mathscr{R}$. Since $\mathbb{Q}\mathfrak{S}_n$ is semisimple as an algebra, we apply the previous paragraph and obtain that $\mathrm{Hom}_{\mathbb{Q}\mathfrak{S}_n}(S^\lambda_\mathbb{Q}, M_\mathbb{Q}(\alpha|\beta))$ has dimension k. Thus there is a non-trivial relation on $\hat{\vartheta}^\mathbb{Q}_{d_0}, \hat{\vartheta}^\mathbb{Q}_{d_1}, \ldots, \hat{\vartheta}^\mathbb{Q}_{d_k}$, which we can write as $\sum_{j=0}^k c_j \hat{\vartheta}^\mathbb{Q}_{d_j} = 0$, where the c_j's are co-prime integers. This yields the non-trivial linear relation $\sum_{j=0}^k c_j^\mathbb{F} \hat{\vartheta}^\mathbb{F}_{d_j} = 0$. Since $\hat{\vartheta}^\mathbb{F}_{d_1}, \hat{\vartheta}^\mathbb{F}_{d_2}, \ldots, \hat{\vartheta}^\mathbb{F}_{d_k}$ are linearly independent, this implies that $\hat{\vartheta}^\mathbb{F}_{d_0}$ lies in the span of $\{\hat{\vartheta}^\mathbb{F}_{d_1}, \hat{\vartheta}^\mathbb{F}_{d_2}, \ldots, \hat{\vartheta}^\mathbb{F}_{d_k}\}$. □

Remark 5.5. As is well-known, $S^\lambda_\mathbb{F}$ lies in a block which is simple if and only if λ is a *p-core partition* (in other words, λ has no rimhook of size p) where p is the characteristic of $\mathbb{F}$.

As an immediate corollary, we have

Corollary 5.6. *The $\mathbb{F}$-span of $\widehat{\Theta}^\mathbb{F}_\mathscr{R}$ has dimension at most $|\mathscr{T}_{\mathrm{sstd}}(\lambda, (\alpha|\beta))|$.*

We now proceed to obtain a sufficient condition for $\widehat{\Theta}^\mathbb{F}_{\mathrm{sstd}}$ to be linearly independent, which by Proposition 5.4 also ensures that it is a basis for the $\mathbb{F}$-span of $\widehat{\Theta}^\mathbb{F}_\mathscr{R}$.

We first introduce a pre-order $\trianglerighteq$ on $\mathscr{T}(\lambda, (\alpha|\beta))$, which induces another on $\mathfrak{S}_n$.

For each $\mathrm{T} \in \mathscr{T}(\lambda, (\alpha|\beta))$, write $C_j(\mathrm{T})$ for the multi-set associated to the jth column of T, i.e.

$$C_j(\mathrm{T}) = \{\mathrm{T}(1,j), \mathrm{T}(2,j), \ldots, \mathrm{T}(r,j)\}$$

where $r = (\lambda')_j$. Recall that we have the total order

$$\mathbb{c}_1 < \mathbb{c}_2 < \cdots < \mathbb{d}_1 < \mathbb{d}_2 < \cdots .$$

Let $\mathrm{T}, \mathrm{T}' \in \mathscr{T}(\lambda, (\alpha|\beta))$. Suppose that $C_j(\mathrm{T}) = \{y_1, \ldots, y_r\}$ and $C_j(\mathrm{T}') = \{z_1, \ldots, z_r\}$, where $y_1, \ldots, y_r, z_1, \ldots, z_r \in \{\mathbb{c}_1, \mathbb{c}_2, \ldots, \mathbb{d}_1, \mathbb{d}_2, \ldots\}$, $y_1 \leq y_2 \leq \cdots \leq y_r$ and $z_1 \leq z_2 \leq \cdots \leq z_r$. Write $C_j(\mathrm{T}) \triangleright C_j(\mathrm{T}')$ if and only if there exists k such that $y_s = z_s$ for all $1 \leq s < k$ and $y_k > z_k$. We write $\mathrm{T} \trianglerighteq \mathrm{T}'$ if and only if $C_j(\mathrm{T}) = C_j(\mathrm{T}')$ for all j, or there exists t such that $C_j(\mathrm{T}) = C_j(\mathrm{T}')$ for all $j < t$ and $C_t(\mathrm{T}) \triangleright C_t(\mathrm{T}')$. It is easy to check that $\trianglerighteq$ is a pre-order on $\mathscr{T}(\lambda, (\alpha|\beta))$ (i.e. it is a reflexive and transitive binary relation on $\mathscr{T}(\lambda, (\alpha|\beta))$). In addition, write $\mathrm{T} \sim \mathrm{T}'$ if and only if $\mathrm{T} \trianglerighteq \mathrm{T}'$ and $\mathrm{T}' \trianglerighteq \mathrm{T}$ (equivalently, $C_j(\mathrm{T}) = C_j(\mathrm{T}')$ for all j), and $\mathrm{T} \triangleright \mathrm{T}'$ if and only if $\mathrm{T} \trianglerighteq \mathrm{T}'$ but $\mathrm{T}' \not\trianglerighteq \mathrm{T}$ (equivalently, there exists t such that $C_j(\mathrm{T}) = C_j(\mathrm{T}')$ for all $j < t$ and $C_t(\mathrm{T}) \triangleright C_t(\mathrm{T}')$).

Recall that $\mathrm{T}_d = d \cdot \mathrm{T}_0$. Let $d, d' \in \mathfrak{S}_n$, and write $d \trianglerighteq d'$, $d \sim d'$ and $d \triangleright d'$ if and only if $\mathrm{T}_d \trianglerighteq \mathrm{T}_{d'}$, $\mathrm{T}_d \sim \mathrm{T}_{d'}$ and $\mathrm{T}_d \triangleright \mathrm{T}_{d'}$ respectively.

Lemma 5.7. *Let λ be a partition of n, $(\alpha|\beta)$ be a bicomposition of n and $d, d' \in \mathfrak{S}_n$.*

(i) *$d \sim d'$ if and only if $d' \in C_{\mathrm{t}_0} d \mathfrak{S}_{\alpha|\beta}$.*

(ii) *If T_d is row semistandard, and $d' \in R_{\mathrm{t}_0} d \mathfrak{S}_{\alpha|\beta}$, then either $\mathrm{T}_{d'} = \mathrm{T}_d$ or $\mathrm{T}_{d'} \triangleright \mathrm{T}_d$.*

(iii) *If $\mathrm{T}_\mathfrak{d} \in \mathscr{T}_{\mathrm{sstd}}(\lambda, (\alpha|\beta))$ and $\mathfrak{a}_{d',\mathfrak{d}} \neq 0$, then $d' \trianglerighteq \mathfrak{d}$.*

Proof. Parts (i) and (ii) are straightforward. For part (iii), if $\mathfrak{a}_{d',\mathfrak{d}} \neq 0$, then there exist $\sigma \in C_{\mathrm{t}_0}$, $\tau \in R_{\mathrm{t}_0}$ and $\xi \in \mathfrak{S}_{\alpha|\beta}$ such that $\sigma d' = \tau \mathfrak{d} \xi$. Thus,

$$d' \sim \sigma d' = \tau \mathfrak{d} \xi \trianglerighteq \mathfrak{d}$$

by parts (i) and (ii) since $\mathrm{T}_{\mathfrak{d}}$ is, in particular, row semistandard. ☐

The following two propositions give some necessary conditions for the set $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ to be linearly independent and a basis for $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$.

Proposition 5.8. *Suppose that* $\hat{\mathfrak{a}}^{\mathbb{F}}_{d,d} \neq 0$ *for all* $d \in \Gamma_{\mathrm{sstd}}$. *Then* $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ *is linearly independent.*

Proof. Suppose the contrary that $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ is linearly dependent, say $\sum_{i=1}^r c_i \hat{\vartheta}^{\mathbb{F}}_{\mathfrak{d}_i} = 0$, where, for each $1 \leq i \leq r$, $\mathfrak{d}_i \in \Gamma_{\mathrm{sstd}}$ and $c_i \in \mathbb{F} \setminus \{0\}$. Since the $\mathrm{T}_{\mathfrak{d}_i}$'s are all column semistandard and distinct, we see that $\mathfrak{d}_i \not\sim \mathfrak{d}_j$ for all $i \neq j$. Thus, relabelling if necessary, we may assume that $\mathfrak{d}_1 \not\trianglerighteq \mathfrak{d}_i$ for all $i \geq 2$. Then

$$0 = \left(\sum_{i=1}^r c_i \hat{\vartheta}^{\mathbb{F}}_{\mathfrak{d}_i}\right)(e_{\mathrm{t}_0}) = \sum_{i=1}^r c_i \sum_{d \in \Gamma} \hat{\mathfrak{a}}^{\mathbb{F}}_{d,\mathfrak{d}_i} (d \otimes \mathbb{1} \otimes \epsilon) = \sum_{d \in \Gamma} \left(\sum_{i=1}^r c_i \hat{\mathfrak{a}}^{\mathbb{F}}_{d,\mathfrak{d}_i}\right)(d \otimes \mathbb{1} \otimes \epsilon),$$

so that $\sum_{i=1}^r c_i \hat{\mathfrak{a}}^{\mathbb{F}}_{d,\mathfrak{d}_i} = 0$ for all $d \in \Gamma$. In particular, $\sum_{i=1}^r c_i \hat{\mathfrak{a}}^{\mathbb{F}}_{\mathfrak{d}_1,\mathfrak{d}_i} = 0$. But for each $2 \leq i \leq r$, we have $\mathfrak{d}_1 \not\trianglerighteq \mathfrak{d}_i$, so that $\mathfrak{a}_{\mathfrak{d}_1,\mathfrak{d}_i} = 0$ by Lemma 5.7(iii), and hence $\hat{\mathfrak{a}}^{\mathbb{F}}_{\mathfrak{d}_1,\mathfrak{d}_i} = 0$. Consequently, $c_1 \hat{\mathfrak{a}}^{\mathbb{F}}_{\mathfrak{d}_1,\mathfrak{d}_1} = 0$, contradicting $c_1 \neq 0$ and $\hat{\mathfrak{a}}^{\mathbb{F}}_{\mathfrak{d}_1,\mathfrak{d}_1} \neq 0$. ☐

The next lemma gives us some idea of what $\hat{\mathfrak{a}}^{\mathbb{F}}_{d,d}$ is for $d \in \Gamma_{\mathrm{sstd}}$.

Lemma 5.9. *Let* $\mathfrak{d} \in \mathscr{R}$ *such that* $\mathrm{T}_{\mathfrak{d}} \in \mathscr{T}_{\mathrm{sstd}}(\lambda, (\alpha|\beta))$. *Then*

$$\mathfrak{a}_{\mathfrak{d},\mathfrak{d}} = |\operatorname{stab}_{C_{\mathrm{t}_0}}(\mathrm{T}_{\mathfrak{d}})|.$$

Proof. Observe that $C_{\mathrm{t}_0} \mathfrak{d} \mathfrak{S}_{\alpha|\beta} \cap R_{\mathrm{t}_0} \mathfrak{d} \mathfrak{S}_{\alpha|\beta} = \mathfrak{d} \mathfrak{S}_{\alpha|\beta}$: if $\sigma \mathfrak{d} = \tau \mathfrak{d} \xi$, where $\sigma \in C_{\mathrm{t}_0}$, $\tau \in R_{\mathrm{t}_0}$ and $\xi \in \mathfrak{S}_{\alpha|\beta}$, then $\mathfrak{d} \sim \tau \mathfrak{d}$ by Lemma 5.7(i) so that $\mathrm{T}_{\tau\mathfrak{d}} \not\triangleright \mathrm{T}_{\mathfrak{d}}$, while $\tau \mathfrak{d} \sim \mathfrak{d}$ only if $\tau \mathfrak{d} \in \mathfrak{d} \mathfrak{S}_{\alpha|\beta}$ by Lemma 5.7(ii). Applying Lemma 4.10(iii) completes the proof since $\operatorname{stab}_{C_{\mathrm{t}_0}}(\mathrm{T}_{\mathfrak{d}}) = C_{\mathrm{t}_0} \cap \mathfrak{d} \mathfrak{S}_{\alpha|\beta} \mathfrak{d}^{-1}$. ☐

Theorem 5.10. *Suppose that* $\mathbb{F}$ *has characteristic* 0, *or* p *with* $p > n$. *Then both* $\overline{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ *and* $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ *are bases for* $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$.

Proof. Under the hypothesis on the characteristic of $\mathbb{F}$, the group algebra $\mathbb{F}\mathfrak{S}_n$ is semisimple, so that every block of $\mathbb{F}\mathfrak{S}_n$ is simple. In addition, $\mathfrak{a}^{\mathbb{F}}_{d,d}$, and hence $\hat{\mathfrak{a}}^{\mathbb{F}}_{d,d}$, are nonzero for all $d \in \Gamma_{\mathrm{sstd}}$ by Lemma 5.9. Thus, $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ is a basis for $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$ by Propositions 5.4 and 5.8. Since $g_d \mid \mathfrak{a}_{d,d}$, we see that $g^{\mathbb{F}}_d \neq 0$, so that $\overline{\Theta}^{\mathbb{F}}_{\mathrm{sstd}} = \{g^{\mathbb{F}}_{\mathfrak{d}} \hat{\vartheta}^{\mathbb{F}}_{\mathfrak{d}} : \hat{\vartheta}^{\mathbb{F}}_{\mathfrak{d}} \in \widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}\}$ is a also a basis for $\mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^\lambda_{\mathbb{F}}, M_{\mathbb{F}}(\alpha|\beta))$. ☐

Theorem 5.11. *Suppose that* $\mathbb{F}$ *has positive characteristic* p. *If no column of* T *has* p *or more nodes of the same colour for every* $\mathrm{T} \in \mathscr{T}_{\mathrm{sstd}}(\lambda, (\alpha|\beta))$, *then both* $\overline{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ *and* $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ *are bases for the* $\mathbb{F}$*-span of* $\widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}}$.

Proof. Let $d \in \Gamma_{\mathrm{sstd}}$. By Lemma 5.9, we see that $\mathfrak{c}^{\mathbb{F}}_{d,d} \neq 0$ if and only if $p \nmid |\mathrm{stab}_{C_{\mathfrak{t}_0}}(\mathrm{T}_d)|$, if and only if no column of T_d has p or more nodes of the same colour. Thus, our hypothesis ensures that $\mathfrak{c}^{\mathbb{F}}_{d,d}$, and hence $\hat{\mathfrak{c}}^{\mathbb{F}}_{d,d}$, are nonzero for all $d \in \Gamma_{\mathrm{sstd}}$. Hence $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ is a basis for the $\mathbb{F}$-span of $\widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}}$ by Propositions 5.4 and 5.8. Since $g_d \mid \mathfrak{c}_{d,d}$, we see that $g^{\mathbb{F}}_d \neq 0$, so that $\overline{\Theta}^{\mathbb{F}}_{\mathrm{sstd}} = g^{\mathbb{F}}_d \widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ is also a basis for $\mathbb{F}$-span of $\widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}}$. □

We end our paper with the following remark and leave the details to the reader.

Remark 5.12. Recall that from Definition 4.3 that the set $\mathscr{R}$ and hence the sets $\overline{\Theta}^{\mathbb{F}}_{\mathscr{R}}$, $\widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}}$, $\overline{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ and $\widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}}$ depend on the fix λ-tableau $\mathfrak{t}_0$ and $(\alpha|\beta)$ we chose. Suppose that we had chosen another λ-tableau $\mathfrak{t}_0'$ and arrived at the sets $\mathscr{R}'$, $\overline{\Theta}^{\mathbb{F}}_{\mathscr{R}'}$, $\widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}'}$, $\overline{\Theta}'^{\mathbb{F}}_{\mathrm{sstd}}$ and $\widehat{\Theta}'^{\mathbb{F}}_{\mathrm{sstd}}$. It is easy to verify that

$$\mathscr{R}' = \pi\mathscr{R}, \qquad \widehat{\Theta}_{\mathscr{R}'} = \widehat{\Theta}_{\mathscr{R}}, \qquad \widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}'} = \widehat{\Theta}^{\mathbb{F}}_{\mathscr{R}}, \qquad \widehat{\Theta}'_{\mathrm{sstd}} = \widehat{\Theta}_{\mathrm{sstd}}, \qquad \widehat{\Theta}'^{\mathbb{F}}_{\mathrm{sstd}} = \widehat{\Theta}^{\mathbb{F}}_{\mathrm{sstd}},$$

where $\pi = \mathfrak{t}_0' \circ (\mathfrak{t}_0)^{-1} \in \mathfrak{S}_n$.

On the other hand, let $\{1, 2, \dots, n\} = \bigcup_{i=1}^{r} A_i \cup \bigcup_{j=1}^{s} B_j$ (disjoint union throughout), with $A_i, B_j \neq \emptyset$ for all i and j. The set $\mathscr{R}$ in Definition 4.3 and the homomorphisms $\bar{\vartheta}_{\mathfrak{d}}$ in Subsection 4.2 and $\hat{\vartheta}_{\mathfrak{d}}$ in this section can be generalised to yield the set $\mathscr{R}_{A|B}$ and homomorphisms $\bar{\vartheta}^{A|B}_{\mathfrak{d}}, \hat{\vartheta}^{A|B}_{\mathfrak{d}} \in \mathrm{Hom}_{\mathbb{Z}\mathfrak{S}_n}(S^{\lambda}_{\mathbb{Z}}, M_{\mathbb{Z}}(A|B))$, and hence $\bar{\vartheta}^{A|B,\mathbb{F}}_{\mathfrak{d}}, \hat{\vartheta}^{A|B,\mathbb{F}}_{\mathfrak{d}} \in \mathrm{Hom}_{\mathbb{F}\mathfrak{S}_n}(S^{\lambda}_{\mathbb{F}}, M_{\mathbb{F}}(A|B))$, for $\mathfrak{d} \in \mathscr{R}_{A|B}$, where

$$M_{\mathcal{O}}(A|B) = \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{A|B}}(\mathcal{O}_{\mathfrak{S}_A} \boxtimes \mathrm{sgn}_{\mathfrak{S}_B}) \qquad (\mathcal{O} \in \{\mathbb{F}, \mathbb{Z}\}),$$

and

$$\mathfrak{S}_A = \prod_{i=1}^{r} \mathfrak{S}_{A_i}, \qquad \mathfrak{S}_B = \prod_{j=1}^{s} \mathfrak{S}_{B_j}, \qquad \mathfrak{S}_{A|B} = \mathfrak{S}_A \mathfrak{S}_B,$$

Therefore, we obtain the respective sets $\overline{\Theta}^{A|B}_{\mathscr{R}}$, $\widehat{\Theta}^{A|B}_{\mathscr{R}}$, $\overline{\Theta}^{A|B,\mathbb{F}}_{\mathscr{R}}$ and $\widehat{\Theta}^{A|B,\mathbb{F}}_{\mathscr{R}}$.

Let $g \in \mathfrak{S}_n$, and let $A_i' = g^{-1}(A_i)$, $B_j' = g^{-1}(B_j)$ for all $i = 1, \dots, r$ and $j = 1, \dots, s$. Then $\{1, 2, \dots, n\} = \bigcup_{i=1}^{r} A_i' \cup \bigcup_{j=1}^{s} B_j'$ (disjoint union throughout), and it is straightforward to show that

$$\begin{aligned} \mathscr{R}_{A'|B'} = \mathscr{R}_{A|B}g, \qquad & \overline{\Theta}^{A'|B'}_{\mathscr{R}} = \varrho_g \circ \overline{\Theta}^{A|B}_{\mathscr{R}}, \qquad & \widehat{\Theta}^{A'|B'}_{\mathscr{R}} = \varrho_g \circ \widehat{\Theta}^{A|B}_{\mathscr{R}}, \\ & \overline{\Theta}^{A'|B',\mathbb{F}}_{\mathscr{R}} = \varrho^{\mathbb{F}}_g \circ \overline{\Theta}^{A|B,\mathbb{F}}_{\mathscr{R}}, \qquad & \widehat{\Theta}^{A'|B',\mathbb{F}}_{\mathscr{R}} = \varrho^{\mathbb{F}}_g \circ \widehat{\Theta}^{A|B,\mathbb{F}}_{\mathscr{R}}, \end{aligned}$$

where ϱ_g is the natural $\mathbb{Z}\mathfrak{S}_n$-module isomorphism $M_{\mathbb{Z}}(A|B) \to M_{\mathbb{Z}}(A'|B')$ defined by $\varrho_g(x \otimes \mathbb{1} \otimes \epsilon) = xg \otimes \mathbb{1} \otimes \epsilon$ for all $x \in \mathfrak{S}_n$ and $\varrho^{\mathbb{F}}_g = 1_{\mathbb{F}} \cdot \varrho_g$.

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