MCSSHANE’S IDENTITY FOR CLASSICAL SCHOTTKY GROUPS

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Abstract. In [15], Greg McShane demonstrated a remarkable identity for the lengths of simple closed geodesics on cusped hyperbolic surfaces. This was generalized by the authors in [19] to hyperbolic cone-surfaces, possibly with cusps and/or geodesic boundary. In this paper, we generalize the identity further to the case of classical Schottky groups. As a consequence, we obtain some surprising new identities in the case of fuchsian Schottky groups. For classical Schottky groups of rank 2, we also give generalizations of the Weierstrass identities, given by McShane in [16].

1. Introduction

In [15] Greg McShane demonstrated a striking identity for the lengths of simple closed geodesics on cusped hyperbolic surfaces. In [19], we gave a generalization of McShane’s identity to hyperbolic cone-surfaces possibly with cusps and/or geodesic boundary (a version for compact hyperbolic surfaces with geodesic boundary was also given independently by M. Mirzakhani in [17]). To state the generalization, we first define two functions $G(x, y, z)$ and $S(x, y, z)$ as follows.

Definition 1.1.

\[ G(x, y, z) := 2 \tanh^{-1} \left( \frac{\sinh(x)}{\cosh(x) + \exp(y + z)} \right), \quad (1) \]

\[ S(x, y, z) := \tanh^{-1} \left( \frac{\sinh(x) \sinh(y)}{\cosh(z) + \cosh(x) \cosh(y)} \right). \quad (2) \]

Note that here, for a complex number $x$, $\tanh^{-1}(x)$ is defined to have imaginary part in $(-\pi/2, \pi/2]$, and that the functions $G(x, y, z)$ and $S(x, y, z)$ are analytic in the arguments $x$, $y$ and $z$.

Recall, as defined in [19], that a \textit{compact hyperbolic cone-surface} is a compact topological surface, $M$, equipped with a hyperbolic cone structure so that each boundary component of $M$ is a smooth geodesic, and there are a finite number of points in the topological interior of $M$, which form the set of cusps and cone points. The \textit{geometric boundary} of $M$ consists of all the cusps, cone points and boundary geodesics. The complement of the geometric boundary in $M$ is its \textit{geometric interior}. Recall also that (a) a geometric boundary component of a compact hyperbolic cone-surface $M$ is either a cusp, a cone point or a boundary geodesic; (b) an interior generalized simple closed geodesic is either a simple closed geodesic in the geometric interior of $M$ or a degenerate one which is the double cover of a simple geodesic.

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arc connecting two angle $\pi$ cone points; (c) a generalized simple closed geodesic is either an interior one as in (b) or a boundary one, that is, a geometric boundary component.

The generalized identity in [19] for compact hyperbolic cone-surfaces can be stated as follows.

**Theorem 1.2.** (Theorem 9.3 [19]) For a compact hyperbolic cone-surface $M$ with all cone angles in $(0, \pi]$, let $\Delta_0, \Delta_1, \cdots, \Delta_m$ be its geometric boundary components with complex lengths $L_0, L_1, \cdots, L_m$ respectively. If $\Delta_0$ is a cone point or a boundary geodesic then

$$
\sum_{\alpha,\beta} G\left(\frac{L_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right) + \sum_{j=1}^{m} \sum_{\beta} S\left(\frac{L_0}{2}, \frac{L_j}{2}, \frac{|\beta|}{2}\right) = \frac{L_0}{2},
$$

where the first sum is taken over all unordered pairs of generalized simple closed geodesics $\alpha, \beta$ on $M$ such that $\alpha, \beta$ bound with $\Delta_0$ an embedded pair of pants on $M$ (note that one of $\alpha, \beta$ might be a geometric boundary component) and the sub-sum in the second sum is taken over all interior simple closed geodesics $\beta$ such that $\beta$ bounds with $\Delta_j$ and $\Delta_0$ an embedded pair of pants on $M$. Furthermore, all the infinite series in (3) converge absolutely.

**Remark 1.3.**

(i) $|\alpha|$ denotes the complex length of the generalized simple closed geodesic $\alpha$, which is the usual hyperbolic length if $\alpha$ is a geodesic, is equal to $i\theta$ if $\alpha$ is a cone point of cone angle $\theta$ and is 0 if $\alpha$ is a cusp point. Similarly, $L_i$ is strictly positive, pure imaginary or 0 depending on whether $\Delta_i$ is a geodesic boundary, cone point or cusp respectively.

(ii) As explained in [19], the above identity unifies all the various known versions of McShane’s identity in the real case by using the complex lengths of the geometric boundary components. A version for surfaces with geodesic boundary and no cone points was also given by Mirzakhani in [17]. See also [1], [2], [4], [5], [6], [16] and [18] for other variations and generalizations of the original identity.

In this paper, motivated by the idea of considering the complexification of the boundary lengths, we extend Theorem 1.2 to classical Schottky groups by analytic continuation. This is possible because firstly, the marked classical Schottky space, appropriately parametrized, is a connected open subset of the parametrization space. Secondly, both sides of the identity (3) are analytic functions of the parameters for the marked classical Schottky space. Finally, all the infinite series on the left hand side of (3) converge absolutely in the marked classical Schottky space. However, we will need to re-interpret $|\alpha|$ as the complex length of $\alpha$ in the formula, and there are some subtleties involved as we will need a specific choice of the half-lengths (recall that the complex length is defined up to multiples of $2\pi i$, and that there are two possible choices for the half lengths, up to multiples of $2\pi i$).

The exact statement (Theorem 3.4) requires a choice of a lift from a representation of the free group $F_n$ into $\text{PSL}(2, \mathbb{C})$ to $\text{SL}(2, \mathbb{C})$ (the exact choice of the lift is not important), a reformulation of Theorem 1.2 to a more algebraic form, precise definitions for a marked classical Schottky group, the marked classical Schottky space, as well as a fuchsian marking for the marked classical Schottky space. The basic
idea is that a fuchsian marking corresponds to a hyperbolic surface $M$ with geodesic boundary. Fixing a boundary component $\Delta_0$ for $M$, we have that the identity (3) holds. Now as we deform away from the fuchsian marking to an arbitrary point in the marked classical Schottky space, the identity continues to hold by analytic continuation, if we can show that the infinite series converge absolutely and uniformly on compact subsets of the marked classical Schottky space. In particular, if we deform to a different fuchsian marked classical Schottky group, the original identity still holds even though the corresponding hyperbolic surface $M'$ may be of a different topological type from the original surface $M$ (or it may be of the same topological type but with a completely different marking). This gives new identities for the surface $M'$. For example, in the case when the Schottky group is of rank 2, $M$ may be the hyperbolic one-holed torus and $M'$ the hyperbolic three-holed sphere (pair of pants), so that we obtain new identities for the hyperbolic pair of pants with geodesic boundary, different from the trivial identity obtained by a direct application of Theorem 1.2.

The rest of this paper is organized as follows. In §2 we give some basic facts about classical Schottky groups and give a precise definition of marked classical Schottky space, as well as a parametrization for the space. In §3 we state and prove the main result, Theorem 3.4. In §4 we state and discuss the Weierstrass identity for marked rank 2 classical Schottky groups (Theorem 4.1). Finally, in §5, we analyze an example to show that our generalization of McShane’s identity to classical Schottky groups does give some surprising new identities for hyperbolic surfaces with geodesic boundary.

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2. Marked classical Schottky groups

In this section we state some basic facts about marked classical Schottky groups. See [11], [12], [13], [7] and [8] for a more complete study of various Schottky spaces. Note that the terminology is not completely standardized, we use the terminology which is best suited for our purposes, in particular, we need to pay special attention to the marking in order to get a clean and precise statement of our result. Hence, we define a marked classical Schottky group and marked classical Schottky space in an analogous way to that of marked hyperbolic structures and Teichmüller space.

Fix $n \geq 2$ and let $F_n = \langle a_1, \ldots, a_n \rangle$ be a free group of rank $n$, where $\{a_1, \ldots, a_n\}$ is a (fixed) distinguished, ordered set of generators for $F_n$, which will provide the marking. Let $\mathbb{C}_\infty$ be the extended complex plane, which we also identify with the Riemann sphere, and also the ideal boundary of $\mathbb{H}^3$.

**Definition 2.1.** A (marked) classical Schottky group (of rank $n$) is a discrete, faithful representation $\rho : F_n \to \text{PSL}(2, \mathbb{C})$ such that there is a region $D \subset \mathbb{C}_\infty$, where $D$ is bounded by $2n$ disjoint geometric circles $C_1, C'_1, \ldots, C_n, C'_n$ in $\mathbb{C}_\infty$, so that, for $i = 1, \ldots, n$, $\rho(a_i)(C_i) = C'_i$, and $\rho(a_i)(D) \cap D = \emptyset$.

Note that the circles $C_i, C'_i$ are not uniquely determined by $\rho$. Also, $\rho(a_i)$ is strictly loxodromic, with an attracting and repelling ideal fixed point, denoted by $\text{Fix}^+ \rho(a_i)$ and $\text{Fix}^- \rho(a_i)$ respectively. The image $\Gamma := \rho(F_n)$ is often referred to in
the literature as a classical Schottky group; it is a Schottky group if we only require $C_1, C'_1, \ldots, C_n, C'_n, i = 1, \ldots, n$ to be disjoint simple closed curves. Two (marked) classical Schottky groups $\rho_1$ and $\rho_2$ are equivalent if the representations are conjugate to each other.

If $\rho$ is a marked classical Schottky group, we denote by $\tilde{\rho}$ any lift of $\rho$ to a representation into $\text{SL}(2, \mathbb{C})$. Note that since $F_n$ is free, $\rho$ can always be lifted and there are $2^n$ possible lifts.

**Remark 2.2.** It was shown by Marden [11] that there exist non-classical Schottky groups for every $n \geq 2$. An explicit example of a non-classical Schottky group was constructed by Yamamoto [22]. On the other hand, Button [7] has proved that all fuchsian Schottky groups are classical (but in general not on every set of generators).

**Definition 2.3.** The space of equivalence classes of (marked) classical Schottky groups is called the *marked classical Schottky space*; we denote it by $S_{\text{mc}}^{\text{alg}}$.

Note that we define (marked) classical Schottky groups in terms of representations rather than the subgroup $\rho(F_n)$ of $\text{PSL}(2, \mathbb{C})$ and the space $S_{\text{alg}}^{\text{mc}}$ as the space of such representations, modulo conjugation. In particular, two representations $\rho_1$ and $\rho_2$ may have the same image in $\text{PSL}(2, \mathbb{C})$ modulo conjugation, but still represent different points of $S_{\text{alg}}^{\text{mc}}$ because of the marking. To simplify notation, we use $\rho$ to denote elements of $S_{\text{alg}}^{\text{mc}}$, instead of $[\rho]$ which is more cumbersome, there should be no confusion as the traces and complex lengths are well-defined on the equivalence classes.

Next we give a natural parametrization of the marked classical Schottky space $S_{\text{alg}}^{\text{mc}}$ by the ideal fixed points and the (trace)$^2$/complex lengths of $\rho(a_i), i = 1, \ldots, n$.

We may normalize $\rho$ by conjugation so that

$$\text{Fix}^{-1} \rho(a_1) = 0, \text{Fix}^{+1} \rho(a_1) = \infty \text{ and } \text{Fix}^{-1} \rho(a_2) = 1.$$  

Then it is not difficult to see that we can parameterize $\rho$ by

$$(\text{Fix}^{+1} \rho(a_2), \text{Fix}^{-1} \rho(a_3), \text{Fix}^{+1} \rho(a_3), \ldots, \text{Fix}^{+1} \rho(a_n); \text{tr}^2 \rho(a_1), \ldots, \text{tr}^2 \rho(a_n))$$

$$\in \mathbb{C}^{2n-3} \times \mathbb{C}^n,$$

or, alternatively, by

$$(\text{Fix}^{+1} \rho(a_2), \text{Fix}^{-1} \rho(a_3), \text{Fix}^{+1} \rho(a_3), \ldots, \text{Fix}^{+1} \rho(a_n); l(\rho(a_1)), \ldots, l(\rho(a_n)))$$

$$\in \mathbb{C}^{2n-3} \times (\mathbb{C}/2\pi i \mathbb{Z})^n,$$

where the complex lengths $l(\rho(a))$ are related to the traces by the formula

$$2 \cosh \frac{l(\rho(a))}{2} = \text{tr} \rho(a),$$

and can be chosen to have positive real part (since all elements are strictly loxodromic). Then the lengths are defined up to multiples of $2\pi i$, and depend only on $|\text{tr} \rho(a)|$ or $\text{tr}^2 \rho(a)$. We shall see later that to define the half-length we will be choosing a lift of the representation and using the negative of the trace on the right hand side of the formula. With this normalized parametrization we have

**Lemma 2.4.** (Maskit [13]) The marked classical Schottky space $S_{\text{alg}}^{\text{mc}}$ is a path connected open subset of $\mathbb{C}^{2n-3} \times (\mathbb{C}/2\pi i \mathbb{Z})^n$. 

Proof. Here we would like to sketch the idea of the proof; see Maskit [13] for a detailed proof. We use the conformal ball model of hyperbolic 3-space $\mathbb{H}^3$. The ideal sphere is then the unit sphere $S_\infty$. Consider a marked classical Schottky group $\rho$ so that $\Gamma = \rho(F_n) \subset \text{Isom}^+(\mathbb{H}^3)$. Then there exists open disks $D_j, D'_j$, $j = 1, \ldots, n$ on the ideal sphere whose boundary circles are denoted $C_j, C'_j$ respectively, such that $\rho(a_j)(C_j) = C'_j$ and $\rho(a_j)(D_j) \cap D'_j = \emptyset$. Now we may find a one parameter family $\rho_t$ of $\rho$ in $S_{\text{alg}}^\text{mc}$, such that the ideal fixed points of all the $\rho_t(a_j)$ are unchanged, but the real part of $l(\rho_t(a_j)) \to \infty$ (that is, the translation length of $\rho_t(a_j)$ approaches $\infty$). We may assume that under the deformation the circles $C_j, C'_j$ shrink towards the respective ideal fixed points of $\rho(a_j)$, so that their sizes become arbitrarily small. This gives us room to continue to deform the marked classical Schottky group in $S_{\text{alg}}^\text{mc}$ continuously to one where the fixed points are in some fixed standard configuration, say, with all of them lying on a circle. Finally, keeping the ideal fixed points fixed, we can deform the lengths to some predetermined quantities, with sufficiently large real part. Since any marked classical Schottky group can be so deformed, the space $S_{\text{alg}}^\text{mc}$ is path connected. That $S_{\text{alg}}^\text{mc}$ is open is easily seen from the definition and parametrization. 

With the parametrization, for each $g \in F_n$, $\rho(g) \in \text{PSL}(2, \mathbb{C})$ is completely determined by the parameters, where $\rho$ is the normalized representation. Hence, for each $g \in F_n$, $\text{tr} \rho(g)$ and $l(\rho(g))$ are analytic functions of the parameters. Furthermore, if we start with a fuchsian Schottky group, we may define all the lengths to be real and positive, and if we extend the definition of the complex lengths by analytic continuation on the space $S_{\text{alg}}^\text{mc}$, then the following proposition states that $\Re(l(\rho(g))) > 0$ for all $\rho \in S_{\text{alg}}^\text{mc}, g \in F_n$.

**Proposition 2.5.** If $\rho_0 \in S_{\text{alg}}^\text{mc}$ is fuchsian, we may define $l(\rho_0(g))$ so that $l(\rho_0(g))$ is positive real for all $g \in F_n$. Then if $l(\rho(g))$ is defined by analytic continuation along a path in the space $S_{\text{alg}}^\text{mc}$, we have $\Re(l(\rho(g))) > 0$ for all $\rho \in S_{\text{alg}}^\text{mc}$ in the path and all $g \in F_n$.

*Proof.* If $\rho_0$ is fuchsian, then it is well known that $|\text{tr} \rho_0(g)| > 2$ for all $g \in F_n$ so that $l(\rho_0(g))$ is positive real. If $\rho \in S_{\text{alg}}^\text{mc}$ and $\rho_t$, $0 \leq t \leq 1$ is a path from $\rho_0$ to $\rho$, then we claim that $\Re(l(\rho_t(g))) > 0$ for all $g \in F_n$, $0 \leq t \leq 1$. Otherwise, we will have that $\Re(\rho_t(g)) = 0$ for some $g \in F_n$, $t \in [0, 1]$, which is impossible as all elements are strictly loxodromic in a classical Schottky group. 

Finally let us say a few words about the fundamental domain in $\mathbb{H}^3$ of a classical Schottky group. Let $\Gamma = \rho(F_n)$ be the image of a classical Schottky group, with the set of disjoint discs $D_j, D'_j$ defined as before. Suppose the circles $C_j, C'_j$ bound respectively geodesic planes $E_j, E'_j$ in $\mathbb{H}^3$. For $j = 1, \ldots, n$ let $H_j$ be the open half space of $\mathbb{H}^3$ bounded by $D_j$ and $E_j$; and similarly for $H'_j$. Then $D := \mathbb{H}^3 - \bigcup_{j=1}^n H_j - \bigcup_{j=1}^n H'_j$ is a fundamental domain in $\mathbb{H}^3$ of $\Gamma$. Note that $D = \overline{D} \cap C_\infty$ is the fundamental domain in the extended complex plane $C_\infty$ of $\Gamma$ as described at the beginning of this section. It is well-known that the quotient hyperbolic 3-manifold $\mathcal{H} = \mathbb{H}^3/\Gamma = D/\Gamma$ is a handlebody of genus $n$. Note that $C_\infty/\Gamma = D/\Gamma$ is a conformal surface of genus $n$ which is the conformal boundary of the handlebody $\mathcal{H}$. 

3. McShane’s identity for Schottky groups

In this section we state and prove our main theorem, Theorem 3.4 below. We start with the following definition.

**Definition 3.1.** A fuchsian marking in $S_{\text{alg}}^\text{mc}$ is an element $\rho_0 \in S_{\text{alg}}^\text{mc}$ which can be conjugated so that $\rho_0(a_i) \in \text{PSL}(2, \mathbb{R})$, for $i = 1, \ldots, n$ (and hence $\rho_0(F_n) \subseteq \text{PSL}(2, \mathbb{R})$), and $C_i, C'_i$ can all be taken to be circles which are orthogonal to $\mathbb{R}$.

For a fuchsian marking $\rho_0$, $\mathbb{H}^2/\rho_0(F_n)$ is a complete hyperbolic surface. Its convex core, $M_0$, is a hyperbolic surface with geodesic boundary, which we call the hyperbolic surface corresponding to the fuchsian marking. Let $\Delta_0, \Delta_1, \ldots, \Delta_m$ be the boundary components of $M_0$. The image $\rho(F_n)$, and hence $F_n$, can be identified with $\pi_1(M_0)$, and if we define an equivalence relation $\sim$ on $F_n$ by $g \sim h$ if $g$ is conjugate to $h$ or $h^{-1}$, then there is a bijection

$$f : F_n/\sim \to C$$

from $F_n/\sim$ to the set $C$ of free homotopy classes of closed curves on $M_0$. Note that there is a unique geodesic representative for each non-trivial element of $C$.

**Definition 3.2.** For a fixed fuchsian marking $\rho_0$, let $M_0$ be the corresponding hyperbolic surface. Let $\Delta_0, \Delta_1, \ldots, \Delta_m$ be the boundary components of $M_0$, and let $[d_i] \in F_n/\sim$, $i = 0, \ldots, m$ be the equivalence class corresponding to the boundary component $\Delta_i$, that is $[d_i] = \Delta_i$.

We define $\mathcal{P}$ to be the set of all unordered pairs $\{[g], [h]\}$ of elements in $F_n/\sim$ such that $f[g]$ and $f[h]$ bound together with $\Delta_0$ an embedded pair of pants in $M_0$ (note that it is possible that $f[g] = \Delta_k$, for some $1 \leq k \leq m$).

For $j = 1, \ldots, m$, we define $\mathcal{B}_j$ to be the set of elements $[g] \in F_n/\sim$ such that $f[g]$ bounds together with $\Delta_0$ and $\Delta_j$ an embedded pair of pants in $M_0$.

We will also need to define the half lengths, for which we need representations into $\text{SL}(2, \mathbb{C})$ instead of $\text{PSL}(2, \mathbb{C})$.

**Definition 3.3.** If $\rho \in S_{\text{alg}}^\text{mc}$ and $\tilde{\rho}$ is a lift of $\rho$ to $\text{SL}(2, \mathbb{C})$, then for an element $g \in F_n$, we define the half length $l(\tilde{\rho}(g))/2 \in \mathbb{C}/2\pi i \mathbb{Z}$ of $\tilde{\rho}(g)$ by

$$\cosh \frac{l(\tilde{\rho}(g))}{2} = -\frac{\text{tr}(\tilde{\rho}(g))}{2}$$

with $\Re(l(\rho(g))) > 0$.

Note that the real part of the half length is just half of the real part of the length, and both are positive, while the above choice fixes the imaginary part, up to multiples of $2\pi i$. The minus sign on the right hand side of (4) is crucial, see Remark 3.5 (iv).

Our main theorem can be stated as follows.

**Theorem 3.4.** Let $\rho \in S_{\text{alg}}^\text{mc}$, and let $\tilde{\rho}$ be any lift of $\rho$ to $\text{SL}(2, \mathbb{C})$. Suppose $\rho_0$ is a fuchsian marking, with corresponding hyperbolic surface $M_0$, with boundary components $\Delta_0, \ldots, \Delta_m$. Let $\mathcal{P}$ and $\mathcal{B}_j$, $j = 1, \ldots, m$ be defined as in Definition
3.2, relative to $M_0$. Then
\[
\sum_{\{(g), (h)\} \in P} G\left(\frac{l(\hat{\rho}(d_0))}{2}, \frac{l(\hat{\rho}(g))}{2}, \frac{l(\hat{\rho}(h))}{2}\right) + \sum_{j=1}^m \sum_{[g] \in B_j} S\left(\frac{l(\hat{\rho}(d_0))}{2}, \frac{l(\hat{\rho}(d_j))}{2}, \frac{l(\hat{\rho}(g))}{2}\right) = \frac{l(\hat{\rho}(d_0))}{2} \mod \pi i. \tag{5}
\]
Moreover, each series on the left-hand side of (5) converges absolutely.

Remark 3.5.

(i) In the case where $\rho = \rho_0$, the above is just a reformulation of Theorem 1.2 for the case of a hyperbolic surface with geodesic boundary components, and is true without the modulo condition. In fact, the lift can be chosen so that the right hand side is real and positive.

(ii) The identity (5) is true only modulo $\pi i$ because we have fixed the choice of the $\tanh^{-1}$ function in the definition of the functions $G(x, y, z)$ and $S(x, y, z)$ (definition 1.1), this may differ from the values obtained by analytic continuation by some multiple of $\pi i$. Indeed, as we will see in Corollary 3.8 and in the example in §5, there is a difference of $2\pi i$ in that example, where $m = 0$, that is, $M_0$ has a single boundary component.

(iii) The result is independent of the lift chosen. This is because if $\hat{\rho}$ and $\bar{\rho}$ are two different lifts of $\rho$, then for each of the summands on the first series, either $\text{tr} \hat{\rho}(g)$, $\text{tr} \hat{\rho}(h)$ and $\text{tr} \hat{\rho}(d_0)$ are all equal to $\text{tr} \bar{\rho}(g)$, $\text{tr} \bar{\rho}(h)$ and $\text{tr} \bar{\rho}(d_0)$ or exactly two of them differ by their signs (and similarly for the summands in the second series). In the latter case, two of the half lengths differ by $\pi i$, but it can be easily checked that both $G(x, y, z)$ and $S(x, y, z)$ remain the same if $\pi i$ is added to two of the arguments.

(iv) The choice of the half length functions given above is not arbitrary but arises from the computation of $G(x, y, z)$ and $S(x, y, z)$ as “gap” functions (this is based on the convention adopted by Fenchel in [9], see [19] for details; see also [10] where Goldman uses a similar convention). Roughly speaking, the relative positions of the axes for $\hat{\rho}(g)$, $\hat{\rho}(h)$ and $\hat{\rho}(d_0)$ are completely determined by their traces. These axes form the non-adjacent sides of a right angled hexagon in $\mathbb{H}^3$ and the half lengths basically arise as the lengths of these sides of the hexagon.

(v) It can be shown that $G(x, y, z)$ and $S(x, y, z)$ can also be expressed as
\[
G(x, y, z) = \log \frac{\exp(x) + \exp(y + z)}{\exp(-x) + \exp(y + z)}, \tag{6}
\]
\[
S(x, y, z) = \frac{1}{2} \log \frac{\cosh(z) + \cosh(x + y)}{\cosh(z) + \cosh(x - y)}, \tag{7}
\]
as used by Mirzakhani in [17] (with different notation), where the function log is in the principal branch, that is, the imaginary parts of its images are in $(-\pi, \pi]$. It would be interesting to see if her results on the Weil-Petersson volumes can be generalized to the classical Schottky space.

Proof. Let $\rho_t$, $0 \leq t \leq 1$ be a deformation from the fuchsian marking to an arbitrary marked classical Schottky group $\rho$, where $\rho_0$ is the fuchsian marking and $\rho_1 = \rho$; this is possible by Lemma 2.4. Let $\hat{\rho}_t$ be a continuous lift of $\rho_t$. We
shall then prove that the series on the left-hand side of (5) converge uniformly on compact subsets of the marked classical Schottky space $S^\text{mc}_{\text{alg}}$. Then each side of (5) is a holomorphic function modulo $\pi i$ on the space $S^\text{mc}_{\text{alg}}$. By Theorem 1.2 the identity (5) holds on the totally real subspace in a neighborhood of $\rho_0$ in $S^\text{mc}_{\text{alg}}$. Note that for this to be true, the correct choice of the half length as given in (4) must be used, see [19] for details. Hence the identity also holds modulo $\pi i$ for each $t \in [0, 1]$ by analytic continuation. This proves the identity for a particular lift; that it holds for all lifts now follow from Remark 3.5(iii).

Given a compact subset $\mathcal{K}$ of $S^\text{mc}_{\text{alg}}$, we have a constant $\kappa > 0$ such that for each $\rho \in \mathcal{K}$, $\rho(F_n)$ has a fundamental domain $D$ in $\mathbb{H}^3$ as described at the end of §2, and the minimum hyperbolic distance between any pair of its bounding geodesic planes $E_1, E_1', \cdots, E_n, E_n'$ is $\geq \kappa$. Then we have the following length estimate lemma for $\rho \in \mathcal{K}$.

**Lemma 3.6.** If $g \in F_n$ is a cyclically reduced word in the set of generators $a_1, a_1^{-1}, \cdots, a_n, a_n^{-1}$ with word length $\|g\|$, then the closed geodesic $\gamma$ which $\rho(g)$ represents in the quotient hyperbolic 3-manifold $D = \mathbb{H}^3/\rho(F_n)$ has hyperbolic length $\geq \kappa\|g\|$. 

**Proof.** Choose in $\mathbb{H}^3$ an arbitrary lift, $\tilde{\gamma}$, of the closed geodesic $\gamma$. Note that $\mathbb{H}^3$ is tiled by the images of a fundamental domain $D$ under the action of elements of $\rho(F_n)$, that is, $\mathbb{H}^3 = \bigcup_{g' \in F_n} \rho(g'(D))$. It can be shown that the line $\tilde{\gamma}$ in $\mathbb{H}^3$ passes through $\|g\|$ successive images of $D$ “periodically” dictated by the word $g$. Thus the hyperbolic length of $\gamma$, which equals the length of the part of $\tilde{\gamma}$ lying in the union of these $\|g\|$ successive images of $D$, is at least $\kappa\|g\|$.

Now we prove the uniform convergence of the first series in (5) for $t \in [0, 1]$. By the above lemma there is a constant $\kappa > 0$ such that for every $t \in [0, 1]$ and every $g \in F_n$, we have $L(\tilde{\rho}_t(g)) \geq \kappa\|g\|$, where $L(\tilde{\rho}_t(g))$ is the hyperbolic length of the closed geodesic that $\tilde{\rho}_t(g)$ represents in the quotient hyperbolic 3-manifold $\mathbb{H}^3/\tilde{\rho}_t(F_n)$, and where $\|g\|$ is the cyclically reduced word length of $g$ in the letters $a_1^{\pm 1}, \cdots, a_n^{\pm 1}$.

Note that the image of the fuchsian marking $\rho_0(F_n) \subset \text{PSL}(2, \mathbb{R})$ has a fundamental domain $D(0)$ in $\mathbb{H}^3$ whose intersection with $\mathbb{H}^2 \subset \mathbb{H}^3$ is a fundamental domain of $\mathbb{H}^2/\rho_0(F_n)$ in $\mathbb{H}^2$. Let $\mathcal{P}$ be the set of unordered pairs of elements $\{[g], [h]\}$ of $F_n/\sim$ as defined in Definition 3.2. Then the pairs $\{[g], [h]\} \in \mathcal{P}$ can be ordered by using the sum of their (cyclically reduced) word lengths $\|g\| + \|h\|$. We have the following version of the Birman-Series result:

**Lemma 3.7.** (c.f. Lemma 2.2 [3]) There exists a polynomial $P(n)$ such that the number of pairs $\{[g], [h]\}$ in $\mathcal{P}$ with $\|g\| + \|h\| = n$ is no greater than $P(n)$.

**Proof.** The proof follows the same line as that used in [3]. We start with the fundamental polygon $D(0) \cap \mathbb{H}^2$, then for each pair $\{[g], [h]\}$ in $\mathcal{P}$ with $\|g\| + \|h\| = n$ we can associate a simple diagram on $D(0) \cap \mathbb{H}^2$ consisting of $n$ disjoint arcs (note that the original Birman-Series’ argument in [3] is for just one simple geodesic, but it works as well here for the pair of disjoint simple closed geodesics on the surface $M_0$ associated with $\{[g], [h]\}$). Conversely, the diagram determines the pair $\{[g], [h]\}$. The number of such simple diagrams is bounded by a polynomial $P(n)$. \hfill \Box
Now, for $t \in [0,1]$ and $[g] \in F_n/\sim$, the real part $\Re(\tilde{\rho}(g))$ of the complex translation length $l(\tilde{\rho}(g))$ is equal to the hyperbolic length $L(\gamma_t(g))$ of the closed geodesic $\gamma_t(g)$, which represents $[g]$ in the quotient hyperbolic manifold $\mathbb{H}^3/\tilde{\rho}(F_n)$. Hence for a pair $\{[g],[h]\}$ in $P$, where $g, h$ are cyclically reduced, we have

$$\left|G\left(\frac{l(\tilde{\rho}(d_0))}{2}, \frac{l(\tilde{\rho}(g))}{2}, \frac{l(\tilde{\rho}(h))}{2}\right)\right| \leq \text{const} \cdot \exp\left(-\frac{L(\gamma_t(g)) + L(\gamma_t(h))}{2}\right) \leq \text{const} \cdot \exp\left(-\frac{\kappa\|g\| + \|h\|}{2}\right), \quad (8)$$

where the first inequality follows from (6) and the fact that $|\log(1 + u)| \leq 2|u|$ for $\log$ in the principal branch and for $u \in \mathbb{C}$ with $|u| \leq \frac{1}{2}$, and the last inequality follows from Lemma 3.6. It now follows Lemma 3.7 that the first series in (5) converges absolutely and uniformly for $t \in [0,1]$. The absolute and uniform convergence for the other series in (5) can be similarly proved.

**Corollary 3.8.** If $m = 0$ in Theorem 3.4 then we have

$$\sum_{([g],[h]) \in P} G\left(\frac{l(\tilde{\rho}(d_0))}{2}, \frac{l(\tilde{\rho}(g))}{2}, \frac{l(\tilde{\rho}(h))}{2}\right) = \frac{l(\tilde{\rho}(d_0))}{2} \mod 2\pi i, \quad (9)$$

(note that here the identity holds modulo $2\pi i$ instead of $\pi i$) and the series converges absolutely.

Note that if $m = 0$, then $d_0$ is actually a commutator so that $\text{tr}(\tilde{\rho}(d_0))$ is independent of the lift $\tilde{\rho}$, hence, the right hand side of (9) is independent of the lift. In fact, it can be shown, with a little bit of extra work, that for the fuchsian marking $\rho_0$, $\text{tr}(\tilde{\rho}(d_0))$ is always strictly negative in this case (see for example [10]). Furthermore, for the more general case where $m \neq 0$, we can always choose a lift $\tilde{\rho}_0$ such that $\text{tr}(\tilde{\rho}_0(d_0))$ is strictly negative.

4. The Weierstrass Indentities

In this section, we consider rank 2 classical Schottky groups $\rho : F_2 \to \text{PSL}(2,\mathbb{C})$. For ease of notation, we denote the marked generators of $F_2$ by $a$ and $b$. Let $\rho_0$ be a fuchsian marking such that the corresponding surface $M_0$ is the one-holed torus with geodesic boundary $\Delta_0$. Let $S$ be the set of non-peripheral simple closed geodesics on $M_0$ and let $w_1, w_2$ and $w_3$ be the Weierstrass points on $M_0$. Then each element of $S$ passes through exactly two of the Weierstrass points, and we define the Weierstrass classes to be the subsets $A_i$, $i = 1, 2, 3$ of $S$ consisting of those geodesics which miss $w_i$. Then $S = \bigsqcup_{i=1}^3 A_i$. Let

$$\tilde{S} := \{[g] \in F_2/\sim | [f[g] \in S]\}, \quad \tilde{A}_i := \{[g] \in F_2/\sim | [f[g] \in A_i]\}$$

be the corresponding sets in $F_2/\sim$. Then $[g] \in \tilde{S}$ if and only if any cyclically reduced representative $g$ forms with another element $h$ a generating set for $F_2$. The set $\tilde{S}$ can be identified with $Q \cup \infty$ by considering the slopes of the corresponding simple closed curves on the torus (without the hole), and the subsets $\tilde{A}_i$, $i = 1, 2, 3$ with the subsets of $Q \cup \infty$ with both numerator and denominator odd, numerator odd and denominator even, and numerator even and denominator odd, respectively; see [20] for details.
We have the following extension of the Weierstrass identities proven by McShane in [16] (see also [19]).

**Theorem 4.1.** For any rank 2 classical Schottky group $\rho \in S \text{alg}_{mc}$, if $A$ is a Weierstrass class, and $\tilde{\rho}$ is a lift of $\rho$, then

$$\sum_{[g] \in A} \tan^{-1}\left(\frac{\cosh \frac{l(\tilde{\rho}(g))}{4}}{\sinh \frac{l(\tilde{\rho}(g))}{2}}\right) = \frac{\pi}{2} \mod \pi,$$

where $[d_0] = [b^{-1}a^{-1}ba] \in F_2/\sim$ corresponds to the boundary $\Delta_0$ of $M_0$.

Note that there are two choices for the quarter-length $\frac{l(\tilde{\rho}(g))}{4}$, we can choose either, but the choice should be the same for all summands. The half length $\frac{l(\tilde{\rho}(g))}{2}$ depends on the choice of the lift $\rho$.

We skip the proof as it is essentially the same as the proof for Theorem 3.4. Here is a geometric interpretation for the above result. First note that the case of the two generator group is very special since for a lift $\tilde{\rho}$, each of $\tilde{\rho}(a)$ and $\tilde{\rho}(a_2)$ can be factored as a product of two involutions (half turns) as follows:

$$\tilde{\rho}(a_1) = -H_3H_2, \quad \tilde{\rho}(a_2) = -H_1H_3$$

where $H_i \in \text{SL}(2, \mathbb{C})$ with $H_i^2 = -I$ and the axis $l_i$ for $H_i$ are such that $l_2$ and $l_3$ are perpendicular to the axis for $\rho(a_1)$ and $l_3$ and $l_1$ are perpendicular to the axis for $\rho(a_2)$ (the minus sign is the convention adopted in Fenchel [9]). Furthermore, in the case of the fuchsian marking $\rho_0$, $l_i$ are lines in $\mathbb{H}^2$ perpendicular to $\mathbb{H}^2$ and passing through lifts of the Weierstrass points $w_i$ respectively. Calling $l_i$ the Weierstrass axes, a deformation $\rho_t$ will correspond to a deformation of the relative positions of the Weierstrass axes. The summands of the left hand side of (10) are then gaps arising from the fixed points of $\rho(g)$ (for certain representatives $g$ of $[g]$) measured against the Weierstrass axis $l$ corresponding to the Weierstrass class $A$.

There are also generalizations of the variations and refinements of McShane’s identity given by Bowditch in [5], Sakuma in [18], Akiyoshi, Miyachi and Sakuma in [1], [2]. In those cases, the variations of McShane’s identity were defined relative to a cusp, these can be generalized to the case of identities relative to a boundary geodesic. We can then study deformations (say in the space of discrete, faithful representations) where the trace of the boundary component remains real and with absolute value > 2. Most of the identities can then be re-interpreted and generalized to this context.

5. An Example: The Three-Holed Sphere

As in the previous section, we consider rank 2 classical Schottky groups $\rho \in S \text{alg}_{mc}$, with the same notation. In particular, $F_2 = \langle a, b \rangle$; $\rho_0$ is a fuchsian marking of $S \text{alg}_{mc}$ where the corresponding hyperbolic surface $M_0$ is a one-holed torus with geodesic boundary $\Delta_0$; and $\mathcal{S} \subset F_2/\sim$ consists of the equivalence classes corresponding to the non-peripheral simple closed geodesics on $M_0$. Let $\rho_1 \in S \text{alg}_{mc}$ correspond to another fuchsian marking, with corresponding hyperbolic surface $M_1$, a pair of pants with geodesic boundary, and let $\mathcal{S}$ be the set of closed geodesics on $M_1$ corresponding to the elements of $\mathcal{S}$. Note that the three boundary geodesics of $M_1$ are elements of $\mathcal{S}$ corresponding to $[a], [b]$ and $[ab]$ (with appropriate orientation). Apart from these, all other elements of $\mathcal{S}$ are non-simple geodesics on $M_1$. 
Let $\delta_0$ be the geodesic on $M_1$ corresponding to the commutator $[d_0] = [b^{-1}a^{-1}ba]$. Note that $\delta_0$ is not a simple geodesic on $M_1$, in fact it has triple self-intersection; see Figure 1.

Let the three geodesic boundary components of $M_1$ be denoted by $\Delta'_0, \Delta'_1, \Delta'_2$ with hyperbolic lengths $l_0 = l(\Delta'_0) > 0, l_1 = l(\Delta'_1) > 0, l_2 = l(\Delta'_2) > 0$ respectively. Then for this hyperbolic surface, we have the following trivial identity

$$G\left(\frac{l_0}{2}, \frac{l_1}{2}, \frac{l_2}{2}\right) + S\left(\frac{l_0}{2}, \frac{l_1}{2}, \frac{l_2}{2}\right) + S\left(\frac{l_0}{2}, \frac{l_2}{2}, \frac{l_1}{2}\right) = \frac{l_0}{2}. \quad (11)$$

There is, however, a non-trivial identity on $M_1$ derived from the fuchsian marking $\rho_0$. Recall that the trace $\text{tr} \rho[d_0]$ is well-defined and independent of the lift of $\rho$, since $d_0$ is a commutator. In fact, for $\rho_0$, we have $\text{tr} \rho_0[d_0] < -2$, and for $\rho_1$, we have $\text{tr} \rho_1[d_0] > 18$ (see Goldman [10] for details where he studied geometric structures arising from 2 generator subgroups of $\text{PSL}(2, \mathbb{C})$ with real character varieties in detail). In particular, the half length $l(\delta_0)/2$ of $\delta_0$ is well-defined up to multiples of $2\pi i$, and $l(\delta_0)/2 = |\delta_0|/2 + \pi i$, where $|\delta_0|$ is the usual hyperbolic length of $\delta_0$ on $M_1$, since $\text{tr} \rho_1[d_0] > 2$ (recall the definition of the half length from (4)). From Corollary 3.8, we have

$$\sum_{\alpha \in \mathcal{S}} G\left(\frac{l(\delta_0)}{2}, \frac{l(\alpha)}{2}, \frac{l(\alpha)}{2}\right) = \frac{l(\delta_0)}{2} \mod 2\pi i, \quad (12)$$

or equivalently,

$$\sum_{\alpha \in \mathcal{S}} 2 \tanh^{-1} \left(\frac{\sinh(l(\delta_0)/2)}{\cosh(l(\delta_0)/2) + \exp l(\alpha)}\right) = \frac{l(\delta_0)}{2} \mod 2\pi i. \quad (13)$$

We see from the second expression that it does not matter which choice of the two half lengths we use for $\alpha$. It turns out that the three summands in the left hand side of (12) or (13) corresponding to the three boundary components of $M_1$ have positive real part and imaginary part equal to $\pi i$ whereas all the other summands
are real and \(<0\). As pointed out earlier, the right hand side is a complex number with positive real part and imaginary part equal to \(\pi i\).

If we define (as in \([20]\)) \(\nu := \log(\delta_0)/2 = \cosh^{-1}(\text{tr}(\rho_1(d_0))/2\), and the functions \(h(x)\) and \(b(x)\) by

\[
h(x) = \frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{x^2}}\right) \quad \text{and} \quad b(x) = \log \left(\frac{1 + (e^{ \nu} - 1) h(x)}{1 + (e^{- \nu} - 1) h(x)}\right),
\]

where the square root is always chosen to have positive real part and we use the principal branch (with imaginary part in \((-\pi, \pi]\)) for the log function, then we can also express (13) as

\[
\sum_{[g] \in \mathcal{S}} b(\text{tr} \rho_1(g)) = \nu \mod 2\pi i. \tag{15}
\]

Remark 5.1. The identity (15) was derived in \([20]\) with a different proof and under more general conditions, and expressed in terms of the \(\mu\)-Markoff map corresponding to the \(\mu\)-Markoff triple \((x, y, z) = (\bar{a} \bar{b} \bar{c}, \bar{b} \bar{c} \bar{a})\). Necessary and sufficient conditions for the identity to hold were also given in \([21]\). For example, the identity still holds if some or all of the boundary components of \(M_1\) degenerate to cusps.

We give a brief description of the geometric interpretation of the above; see \([23]\) or \([20]\) for more details. For each \([g] \in \mathcal{S}\), we may choose representatives \(g_1\) and \(g_2\) such that \(g_1 g_2 = d_0\). Then each of the summands in the various versions of the identity above can be interpreted as gaps from the attractive fixed point of \(\rho_1(g_1)\) to the repelling fixed point of \(\rho_1(g_2)\) measured along the (directed) axis of \(\rho_1(d_0)\). In the cases where \([g] \in \mathcal{S}\) corresponds to the three boundary components of \(M_1\) (that is, \([a]\), \([b]\) and \([ab]\)), the above fixed points lie on the two different intervals of \(\mathbb{R} \cup \infty\) separated by the fixed points of \(\rho_1(d_0)\), which is why the summand has imaginary part \(\pi i\) in these three cases. For the other summands, the fixed points lie on the same interval and the summands are real. See Figure 2 where we have \(\rho_1(a) = A, \rho_1(b) = B\), where \(A, B \in \text{PSL}(2, \mathbb{R})\), and we use the notation \(\bar{A} := A^{-1}, \bar{B} := B^{-1}\). The picture is normalized so that the fixed points of the commutator \(\bar{B} \bar{A} \bar{B} \bar{A} \bar{B}\) are 0 and \(\infty\). In the figure, the gap arising from \([a] \in \mathcal{S}\) corresponding to one of the boundary components of \(M_1\) is measured by dropping perpendiculars from \(\text{Fix}^+(A)\) and \(\text{Fix}^+(\bar{B} \bar{A} \bar{B})\) to the axis \([0, \infty]\) of \(\bar{B} \bar{A} \bar{B}\); hence this gap has positive real part and imaginary part \(\pi i\). Note that \((\bar{B} \bar{A} \bar{B})^{-1} A = \rho_1(d_0)\). Similarly the other two middle-sized dotted semi-circles in the figure show the gaps arising from \([b]\) and \([ab]\) corresponding to the other two boundary components. Gaps for the other elements of \(\mathcal{S}\) can be similarly obtained by a recursive process using the Farey construction of the rationals; all these gaps will be real and negative (they are represented by the solid semi-circles in the figure).

Finally, we note that more generally, if the rank \(n \geq 3\), it is possible to have two fuchsian markings \(\rho_0\) and \(\rho_1\) such that the corresponding hyperbolic surfaces \(M_0\) and \(M_1\) are homeomorphic but the markings are different. Then \(\rho_0\) will induce identities for \(M_1\) which are different from those obtained from \(\rho_1\); for example the gaps may be measured against a geodesic on \(M_1\) which is not necessarily a boundary geodesic, or even simple.
\[ \text{Fix}^+(AB) \quad \text{Fix}^+(A) \quad \text{Fix}^+(BA) \quad \text{Fix}^+(B) \quad \text{Fix}^+(BAB) \]

\[ \bar{A} = A^{-1}, \quad \bar{B} = B^{-1} \]

**Figure 2.** Gaps for \( \mathcal{M}_1 \)

**References**


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