A polynomial-time inexact primal-dual infeasible path-following algorithm for convex quadratic SDP

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Abstract

Convex quadratic semidefinite programming (QSDP) has been widely applied in solving engineering and scientific problems such as nearest correlation problems and nearest Euclidean distance matrix problems. In this paper, we study an inexact primal-dual infeasible path-following algorithm for QSDP problems of the form: $\min_X\{\frac{1}{2}X\bullet\mathcal{Q}(X)+C\bullet X : \mathcal{A}(X)=b,\ X\succeq 0\}$, where $\mathcal{Q}$ is a self-adjoint positive semidefinite linear operator on $\mathcal{S}^n$, $b\in\mathbb{R}^m$, and $\mathcal{A}$ is a linear map from $\mathcal{S}^n$ to $\mathbb{R}^m$. This algorithm is designed for the purpose of using an iterative solver to compute an approximate search direction at each iteration. It does not require feasibility to be maintained even if some iterates happened to be feasible. By imposing mild conditions on the inexactness of the computed directions, we show that the algorithm can find an ϵ-solution in $O(n^2\ln(1/\epsilon))$ iterations.

keywords: semidefinite programming, semidefinite least squares, infeasible interior point method, inexact search direction, polynomial complexity

1 Introduction

We consider the following linearly constrained convex quadratic semidefinite programming (QSDP) problem defined in the vector space of $n\times n$ real symmetric matrices

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$\mathcal{S}^n$ endowed with the inner product $\langle A,\, B\rangle = A \bullet B = \mathrm{Tr}(AB)$:

$$\begin{aligned} (P) \quad & \min \; f(X) := \tfrac{1}{2} X \bullet \mathcal{Q}(X) + C \bullet X \\ & \text{s.t.} \quad A_i \bullet X = b_i, \; i = 1, \cdots, m \\ & \qquad\; X \succeq 0, \end{aligned} \tag{1.1}$$

where $\mathcal{Q} : \mathcal{S}^n \to \mathcal{S}^n$ is a given self-adjoint positive semidefinite linear operator. Here, $A_i, C \in \mathcal{S}^n, b \in \mathbb{R}^m$ are given data and $X \succeq 0$ ($X \succ 0$) indicates that X is in $\mathcal{S}^n_+$ ($\mathcal{S}^n_{++}$). The set $\mathcal{S}^n_+$ ($\mathcal{S}^n_{++}$) denotes the set of positive semidefinite (definite) matrices in $\mathcal{S}^n$. In addition, we assume that $\{A_i \mid i = 1, \ldots, m\}$ is linearly independent. The dual problem of (P) is given as follows:

$$\begin{aligned} (D) \quad & \max \; -\tfrac{1}{2} X \bullet \mathcal{Q}(X) + b^T y \\ & \text{s.t.} \quad \textstyle\sum_{i=1}^m y_i A_i + Z = \nabla f(X) = \mathcal{Q}(X) + C \\ & \qquad Z \succeq 0. \end{aligned} \tag{1.2}$$

The problem (P) includes linear SDP as a special case when $\mathcal{Q} = 0$. It also includes the following linearly constrained convex quadratic programming (LCCQP) [8]:

$$\min \left\{ \frac{1}{2} x^T Q x + c^T x : \; Ax = b, x \in \mathbb{R}^n_+ \right\},$$

where Q is a given positive semidefinite matrix.

A recent application of QSDP is the nearest correlation matrix problem [3]. QSDP also arises in nearest Euclidean distance matrix problems [1] and other matrix least square problems [9]. Many problems in metric embeddings, covariance estimations, and molecular conformations can also be formulated as QSDP, see for example [5] and [13].

We use the following notation and terminology. Let $\bar{n} = n(n+1)/2$. We define the linear map $\mathbf{svec} : \mathcal{S}^n \to \mathbb{R}^{\bar{n}}$ by:

$$\mathbf{svec}(X) := (x_{11}, \sqrt{2}x_{21}, \ldots, \sqrt{2}x_{n1}, x_{22}, \sqrt{2}x_{32}, \ldots, \sqrt{2}x_{n2}, \ldots, x_{nn})^T.$$

The inverse map of $\mathbf{svec}$ is denoted by $\mathbf{smat}$. The matrix representation of $\mathcal{Q}$ in the standard basis of $\mathcal{S}^n$ is the unique matrix $Q \in \mathcal{S}^{\bar{n}}_+$ that satisfies $\mathbf{svec}(\mathcal{Q}(X)) = Q(\mathbf{svec}X)$ for all $X \in \mathcal{S}^n$. Also, let $\mathbf{A}^T = [\mathbf{svec}A_1 \, \mathbf{svec}A_2 \cdots \mathbf{svec}A_m]$, the matrix representations of $A_i \bullet X$ ($i = 1, \cdots, m$) and $\sum_{i=1}^m y_i A_i$ can be written as $\mathbf{A}(\mathbf{svec}X)$ and $\mathbf{A}^T y$ respectively. Note that $\mathbf{A}$ has full row rank and hence $\mathbf{A}\mathbf{A}^T$ is non-singular. The pseudo inverse of $\mathbf{A}$ is defined as $\mathbf{A}^+ = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$. We use $\|\cdot\|$ to denote the Frobenius norm for a matrix or Euclidean norm for a vector, and $\|\cdot\|_2$ to denote the spectral norm of a matrix or the induced norm of a linear operator. For an $n \times n$ matrix M, we ordered its eigenvalues $\lambda_i(M)$ as follows: $\mathrm{Re}\lambda_1(M) \leq \ldots \leq \mathrm{Re}\lambda_n(M)$.

The perturbed Karush-Kuhn-Tucker (KKT) optimality conditions for the problems (P) and (D) are as follows:

$$\begin{pmatrix} -\mathbf{svec}\nabla f(X) + \mathbf{A}^T y + \mathbf{svec} Z \\ \mathbf{A}(\mathbf{svec} X) - b \\ XZ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \nu I \end{pmatrix}, \quad X, Z \succeq 0, \tag{1.3}$$

where $\nu \geq 0$ is a given parameter that is to be driven to zero explicitly. Note that when $\nu = 0$, (1.3) gives the optimal conditions for (P) and (D). As the system (1.3) has more independent equations than unknowns due to the fact that XZ is usually nonsymmetric, the last equation $XZ = \nu I$ is usually symmetrized to $H_P(XZ) = \nu I$, where for a given positive definite matrix P, $H_P : \mathbb{R}^{n\times n} \to \mathcal{S}^n$ is the following symmetrization operator [20] defined by

$$H_P(M) \;:=\; \frac{1}{2}\Big[PMP^{-1} + (PMP^{-1})^T\Big].$$

In [20], P is chosen to be in the class $\mathcal{C}(X,Z) := \{P \in \mathcal{S}^n_{++} \mid PXP \text{ and } P^{-1}ZP^{-1} \text{ commutes}\}$. This class includes the common choices: $P = Z^{1/2}$, $P = X^{-1/2}$, and $P = W^{-1/2}$ where W is the Nesterov-Todd (NT) scaling matrix satisfying $WZW = X$ [14]. It has been shown in [20] that for $X, Z \in \mathcal{S}^n_{++}$ and $P \in \mathcal{C}(X,Z)$, $H_P(XZ) = \nu I$ if and only if $XZ = \nu I$.

In this paper, we choose P to be the NT scaling matrix rather than any $P \in \mathcal{C}(X,Z)$ as considered in [20]. The main reasons for considering only the NT scaling matrix are that it simplifies the complexity analysis and also gives the best iteration complexity. In addition, it is employed in practical computations since it has certain desirable properties that allow one to design efficient preconditioners for the augmented system (3.5a) for computing search directions; see [16] for details.

Primal-dual path-following interior-point methods (IPM) are known to be highly efficient methods for solving linear SDP problems, both in computation [15] and in theoretical complexity [11, 20]. The earliest extension of standard primal-dual path-following algorithms to solve QSDP was done in [1] where for each iteration, a linear system of dimension $m + p$ must be solved directly, say by Cholesky decomposition. Here, p is the rank of Q, and $p = \bar{n}$ if Q is nonsingular. For an ordinary desktop PC, this direct approach can only solve small size problems with n less than a hundred due to the prohibitive computational cost and huge memory requirement when n is large.

In recent applications such as the nearest Euclidean distance matrix completion problems arising from molecular conformation or senor network localization, there is an increasing demand for methods that can handle QSDP where n or m is large. This motivated us to pursue the idea of solving the large linear system inexactly by an iterative solver to overcome the bottleneck mentioned in the last paragraph. Infeasible primal-dual path-following algorithms using inexact search directions have been investigated extensively in LP, linear SDP, and more generally monotone linear complementarity problems; see [2], [7], [12] and [18]. For linear SDP, an inexact infeasible interior-point algorithm was introduced by Kojima et al. in [6] wherein the algorithm only allowed inexactness in the component corresponding to the complementarity equation (the third equation in (1.3)). Subsequently, Zhou and Toh [19] developed an infeasible inexact path-following algorithm which allowed inexactness in the primal and dual feasibilities, and complementarity equations. Furthermore, primal and dual feasibilities need not be maintained even if some iterates happen to lie in the feasible region. In [19], it is proved that the algorithm needs at most $O(n^2 \ln(1/\epsilon))$ iterations to compute an ϵ-optimal solution.

Our interest in this paper is to extend the inexact primal-dual infeasible path-following algorithm in [19] to the case of QSDP. We will focus on establishing the polynomial iteration complexity of the algorithm. In particular, we show that the algorithm needs at most $O(n^2 \ln(1/\epsilon))$ iterations to compute an ϵ-optimal solution for (P) and (D). This complexity result is the same as that established for a linear SDP in [19]. The complexity analysis of our proposed algorithm is similar to the case of a linear SDP in [19]. But there is a major difference in that we always have to consider the effect of the quadratic term in the objective function of QSDP. In particular, Lemma 3.4 shows that the complexity bound we obtained is dependent on $\|\mathcal{Q}\|_2$. We hope that the theoretical framework we developed here for QSDP can lead to further development of inexact primal-dual infeasible path-following methods for broader classes of SDP problems such as those with an objective function $f(X)$ in (P) that is convex with a Lipschitz continuous gradient but not necessarily quadratic.

We should point out that the numerical implementation and evaluation of our proposed inexact algorithm for QSDP has been thoroughly studied in [16] and [17].

The rest of this paper is organized as follows. In the next section, we define the infeasible central path and its corresponding neighborhood. In addition, we also establish some key lemmas that are needed for subsequent complexity analysis. In section 3, we discuss the computation of inexact search directions. We also present our inexact primal-dual infeasible path-following algorithm and establish a polynomial complexity result for this algorithm. In section 4, we give detailed proofs on the polynomial complexity result.

Throughout the paper, we made the following assumption.

Assumption 1. *Problems (P) and (D) are strictly feasible. We say that (P) and (D) are (strictly) feasible if there exists* (X, y, Z) *satisfying the linear constraints in (1.3) and* $X, Z \succeq 0$ $(X, Z \succ 0)$.

2 An infeasible central path and its neighborhood

Let $L = \|\mathcal{Q}\|_2$. Note that L is a Lipschitz constant of the gradient of $f(X)$ defined in (P), i.e.,

$$\|\nabla f(X) - \nabla f(Y)\| = \|\mathcal{Q}(X) - \mathcal{Q}(Y)\| \leq L\|X - Y\|. \tag{2.1}$$

Let (X_0, y_0, Z_0) be an initial point such that

$$X_0 = Z_0 = \rho I, \tag{2.2}$$

where $\rho > 0$ is a given constant. For given positive constants $\gamma_p \leq \gamma_d$ such that $\gamma_d + L\gamma_p \in (0, 1)$, the constant ρ is chosen to be sufficiently large so that for some solution (X_*, y_*, Z_*) to (P) and (D), the following conditions hold:

$$(1 - \gamma_p)X_0 \succ X_* \succeq 0, \quad (1 - (\gamma_d + L\gamma_p))Z_0 \succ Z_* \succeq 0, \tag{2.3}$$

$$\mathrm{Tr}(X_*) + \mathrm{Tr}(Z_*) \;\leq\; n\rho. \tag{2.4}$$

Remark. Under the condition $\gamma_d + L\gamma_p < 1$, γ_p could be close to 0 for a large L. Without loss of generality, we may always assume $L \leq 1$. This can be easily achieved by scaling $f(X)$ with a proper constant. In particular, for the case where $\|Q\|_2 > 1$, we may consider the following pair of scaled primal and dual problems instead:

$$(P') \quad \min \left\{ \frac{1}{2} X \bullet \widehat{\mathcal{Q}}(X) + \widehat{C} \bullet X \mid \mathbf{A}(\mathbf{svec}X) = b,\ X \succeq 0 \right\},$$

$$(D') \quad \max \left\{ -\frac{1}{2} X \bullet \widehat{\mathcal{Q}}(X) + b^T y \mid \mathbf{A}^T y + Z = \widehat{\mathcal{Q}}(X) + \widehat{C},\ Z \succeq 0 \right\},$$

where $\widehat{\mathcal{Q}} = \mathcal{Q}/\|\mathcal{Q}\|_2$ and $\widehat{C} = C/\|\mathcal{Q}\|_2$.

We define

$$\mu_0 \;=\; X_0 \bullet Z_0/n \;=\rho^2, \tag{2.5}$$

$$R_0^p \;=\; \mathbf{A}(\mathbf{svec}X_0) - b, \tag{2.6}$$

$$\mathbf{svec}R_0^d \;=\; -\mathbf{svec}\nabla f(X_0) + \mathbf{A}^T y_0 + \mathbf{svec}Z_0. \tag{2.7}$$

For $\theta, \nu \in (0, 1]$, the following infeasible KKT system has a unique solution under Assumption 1:

$$\begin{pmatrix} -\mathbf{svec}\nabla f(X) + \mathbf{A}^T y + \mathbf{svec}Z \\ \mathbf{A}(\mathbf{svec}X) - b \\ H_P(XZ) \end{pmatrix} = \begin{pmatrix} \theta \mathbf{svec}R_0^d \\ \theta R_0^p \\ \nu\mu_0 I \end{pmatrix}, \quad X, Z \succ 0. \tag{2.8}$$

Define the infeasible central path as:

$$\mathcal{P} = \left\{(\theta, \nu, X, y, Z) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times S^n_{++} \times \mathbb{R}^m \times S^n_{++} \text{ such that (2.8) holds}\right\}.$$

The primary idea of a primal-dual infeasible path-following algorithm is to generate a sequence of points (X^k, y^k, Z^k) such that $(\theta^k, \nu^k, X^k, y^k, Z^k) \in \mathcal{P}$ and (X^k, y^k, Z^k) converges to a solution of (P) and (D) when θ^k and ν^k are driven to 0. In practice of course, the points are never exactly on the central path $\mathcal{P}$ but lie in some neighborhood of $\mathcal{P}$. In our inexact primal-dual infeasible path-following algorithm, we consider the following neighborhood of $\mathcal{P}$. Choose a constant $\gamma \in (0, 1)$ in addition to γ_p and γ_d, we define the neighborhood to be:

$$\mathcal{N} = \left\{ \begin{array}{l} (\theta, \nu, X, y, Z) \in (0,1] \times (0,1] \times \mathcal{S}^n_{++} \times \mathbb{R}^m \times \mathcal{S}^n_{++} : \theta \le \nu, \\ -\mathbf{svec}\nabla f(X) + \mathbf{A}^T y + \mathbf{svec}Z = \theta(\mathbf{svec}R^d_0 + \xi^d),\ \|\xi^d\| \le \gamma_d \rho, \\ \mathbf{A}(\mathbf{svec}X) - b = \theta(R^p_0 + \xi^p),\ \|\mathbf{A}^+\xi^p\| \le \gamma_p \rho, \\ (1-\gamma)\nu\mu_0 \le \lambda_{\min}(XZ) \le \lambda_{\max}(XZ) \le (1+\gamma)\nu\mu_0 \end{array} \right\}.$$

Let $\theta_0 = \nu_0 = 1$. It follows from (2.2) that $(\theta_0, \nu_0, X_0, y_0, Z_0) \in \mathcal{N}$. It is easy to show that if $(\theta, \nu, X, y, Z) \in \mathcal{N}$ and $P \in \mathcal{C}(X, Z)$, then $H_P(XZ) = PXZP^{-1}$ is symmetric and has the same set of eigenvalues as XZ. From the definition of $\mathcal{N}$, it is easy to see that we have

$$(1-\gamma)\nu\mu_0 I \preceq H_P(XZ) \preceq (1+\gamma)\nu\mu_0 I \tag{2.9}$$

$$(1-\gamma)\nu\mu_0 \ \le\ X \bullet Z/n \ \le\ (1+\gamma)\nu\mu_0. \tag{2.10}$$

Next, we present two lemmas that are needed for the iteration complexity analysis in section 3.

Lemma 2.1. *For any r_p and r_d satisfying $\|r_d\| \le \gamma_d\rho$ and $\|\mathbf{A}^+ r_p\| \le \gamma_p\rho$, there exists $(\widetilde{X}, \widetilde{y}, \widetilde{Z})$ that satisfies the following conditions:*

$$-\mathbf{svec}\nabla f(\widetilde{X}) + \mathbf{A}^T\widetilde{y} + \mathbf{svec}\widetilde{Z} \ = \ \mathbf{svec}R^d_0 + r_d\ , \tag{2.11}$$

$$\mathbf{A}(\mathbf{svec}\widetilde{X}) - b \ = \ R^p_0 + r_p\ , \tag{2.12}$$

$$(1-\gamma_p)\rho I \preceq \ \widetilde{X} \ \preceq (1+\gamma_p)\rho I\ , \tag{2.13}$$

$$[1-(\gamma_d + L\gamma_p)]\rho I \preceq \ \widetilde{Z} \ \preceq [1+(\gamma_d + L\gamma_p)]\rho I\ . \tag{2.14}$$

Proof. Let

$$\begin{aligned} \mathbf{svec}\widetilde{X} &= \mathbf{svec}X_0 + \mathbf{A}^+ r_p\ , \\ \widetilde{y} &= y_0\ , \end{aligned}$$

$$\mathbf{svec}\widetilde{Z} \;=\; \mathbf{svec}Z_0 + r_d + Q(\mathbf{svec}\widetilde{X}) - Q(\mathbf{svec}X_0)\;,$$

(2.11)–(2.13) are readily shown. To show (2.14), we only need to establish the following inequality:

$$\|r_d + Q(\mathbf{svec}\widetilde{X}) - Q(\mathbf{svec}X_0)\| \;\le\; \|r_d\| + \|Q(\mathbf{svec}\widetilde{X} - \mathbf{svec}X_0)\| \;\le\; (\gamma_d + L\gamma_p)\rho.$$

□

Lemma 2.2. *Given the initial conditions (2.2), (2.3) and (2.4), for any* $(\theta,\nu,X,y,Z) \in \mathcal{N}$*, we have*

$$\theta \mathrm{Tr}(X) \le \frac{6\nu\rho n}{1-(\gamma_d + L\gamma_p)}, \quad \theta\mathrm{Tr}(Z) \le \frac{6\nu\rho n}{1-\gamma_p}.$$

Proof. This proof is adapted from that for Lemma 2 in [19]. For $(\theta,\nu,X,y,Z) \in \mathcal{N}$, we have

$$-\mathbf{svec}\nabla f(X) + \mathbf{A}^T y + \mathbf{svec}Z = \theta(\mathbf{svec}R_0^d + r_d), \quad \|r_d\| \le \gamma_d\rho, \tag{2.15}$$

$$\mathbf{A}(\mathbf{svec}X) - b = \theta(R_0^p + r_p), \quad \|\mathbf{A}^+ r_p\| \le \gamma_p\rho. \tag{2.16}$$

By Lemma 2.1, there exists $(\widetilde{X}, \widetilde{y}, \widetilde{Z})$ satisfies conditions (2.11)–(2.14). Also, a solution (X_*, y_*, Z_*) to (P) and (D) satisfies the following equations:

$$\mathbf{A}(\mathbf{svec}X_*) - b = 0,$$
$$-\mathbf{svec}\nabla f(X_*) + \mathbf{A}^T y_* + \mathbf{svec}Z_* = 0.$$

Let

$$\widehat{X} = (1-\theta)X_* + \theta\widetilde{X} - X, \quad \widehat{y} = (1-\theta)y_* + \theta\widetilde{y} - y, \quad \widehat{Z} = (1-\theta)Z_* + \theta\widetilde{Z} - Z.$$

Then we have

$$\mathbf{A}(\mathbf{svec}\widehat{X}) = 0, \quad \mathbf{A}^T(\widehat{y}) + \mathbf{svec}\widehat{Z} = Q\,\mathbf{svec}\widehat{X}.$$

Hence $\langle \widehat{X},\, \widehat{Z}\rangle = \langle \widehat{X},\, \mathcal{Q}(\widehat{X})\rangle$. Together with the fact that $\mathcal{Q}$ is positive semidefinite, we have

$$\begin{aligned}
&\big\langle (1-\theta)X_* + \theta\widetilde{X},\, Z\big\rangle + \big\langle X,\, (1-\theta)Z_* + \theta\widetilde{Z}\big\rangle \\
=\;& \big\langle (1-\theta)X_* + \theta\widetilde{X},\, (1-\theta)Z_* + \theta\widetilde{Z}\big\rangle + \big\langle X,\, Z\big\rangle - \big\langle \widehat{X},\, \mathcal{Q}(\widehat{X})\big\rangle \\
\le\;& \big\langle (1-\theta)X_* + \theta\widetilde{X},\, (1-\theta)Z_* + \theta\widetilde{Z}\big\rangle + \big\langle X,\, Z\big\rangle.
\end{aligned} \tag{2.17}$$

By using (2.4), (2.10), (2.13), (2.14), (2.17), and the fact that $X_* \bullet Z_* = 0$, $X_* \bullet Z$, $X \bullet Z_* \ge 0$, we have that

$$\theta\rho[(1-(\gamma_d + L\gamma_p))I \bullet X + (1-\gamma_p)I \bullet Z] \;\le\; \theta(\widetilde{Z} \bullet X + \widetilde{X} \bullet Z)$$

$$\leq \left\langle (1-\theta)X_* + \theta\widetilde{X},\ Z \right\rangle + \left\langle X,\ (1-\theta)Z_* + \theta\widetilde{Z} \right\rangle$$
$$\leq \left\langle (1-\theta)X_* + \theta\widetilde{X},\ (1-\theta)Z_* + \theta\widetilde{Z} \right\rangle + \left\langle X,\ Z \right\rangle$$
$$\leq \theta(1-\theta)(X_* \bullet \widetilde{Z} + \widetilde{X} \bullet Z_*) + \theta^2 \widetilde{X} \bullet \widetilde{Z} + X \bullet Z$$
$$\leq \theta(1-\theta)(1+\gamma_d + L\gamma_p)\rho(X_* \bullet I + I \bullet Z_*) + \theta^2(1+\gamma_p)(1+\gamma_d + L\gamma_p)\rho^2 n + (1+\gamma)\nu\mu_0 n$$
$$\leq 6\nu\rho^2 n.$$

From here, the required results follow. □

Remark. $\{(X, y, Z) \mid (\theta, \nu, X, y, Z) \in \mathcal{N}\}$ is bounded if $\theta = \nu$, since from Lemma 2.1 we have $\|X\| \leq \text{Tr}(X) \leq O(\rho n)$ and $\|Z\| \leq \text{Tr}(Z) \leq O(\rho n)$. Suppose we generate a sequence $\{(\theta_k, \nu_k, X_k, y_k, Z_k)\} \in \mathcal{N}$ such that

$$\nu_k \geq \theta_k,\ \forall k, \ \text{ and } 1 = \nu_0 \geq \nu_k \geq \nu_{k+1} \geq 0.$$

If $\nu_k \to 0$ as $k \to \infty$, then any limit point of the sequence $\{X_k, y_k, Z_k\}$ is a solution of (P) and (D). In particular, if $\theta_k = \nu_k$, then the sequence $\{X_k, Z_k\}$ is also bounded.

3 An inexact infeasible interior point algorithm

Let $\eta_1, \eta_2 \in (0, 1]$ be given constants such that $\eta_1 \geq \eta_2$. Given a current iterate $(\theta_k, \nu_k, X_k, y_k, Z_k) \in \mathcal{N}$, we want to construct a new iterate which remains in $\mathcal{N}$ with respect to smaller θ and ν. To this end, we consider the search direction $(\Delta X_k, \Delta y_k, \Delta Z_k)$ determined by the following linear system:

$$\begin{pmatrix} -Q & \mathbf{A}^T & I \\ \mathbf{A} & 0 & 0 \\ E_k & 0 & F_k \end{pmatrix} \begin{pmatrix} \mathbf{svec}\Delta X_k \\ \Delta y_k \\ \mathbf{svec}\Delta Z_k \end{pmatrix} = \begin{pmatrix} -\eta_1(\mathbf{svec}R_k^d + r_k^d) \\ -\eta_1(R_k^p + r_k^p) \\ \mathbf{svec}R_k^c + r_k^c \end{pmatrix}, \tag{3.1}$$

where for $P_k = W_k^{-1/2}$ (W_k is the NT scaling matrix satisfying $W_k Z_k W_k = X_k$),

$$E_k = P_k \circledast P_k^{-1} Z_k, \quad F_k = P_k^{-1} \circledast P_k X_k$$
$$\mathbf{svec}R_k^d = -\mathbf{svec}\nabla f(X_k) + \mathbf{A}^T y_k + \mathbf{svec}Z_k, \quad R_k^p = \mathbf{A}(\mathbf{svec}X_k) - b$$
$$R_k^c = (1-\eta_2)\nu_k\mu_0 I - H_{P_k}(X_k Z_k).$$

Here $A \circledast B$ denotes the symmetric Kronecker product of any two $n \times n$ matrices A and B, and for any $X \in \mathcal{S}^n$, it is defined by

$$(A \circledast B)\mathbf{svec}(X) := \frac{1}{2}\mathbf{svec}(AXB^T + BXA^T). \tag{3.2}$$

We refer the reader to the appendix of [14] for some of its properties. The last equation of (3.1) is equivalent to

$$H_{P_k}(X_k Z_k + \Delta X_k Z_k + X_k \Delta Z_k) = (1-\eta_2)\nu\mu_0 I + \mathbf{smat} r_k^c. \tag{3.3}$$

The search direction $(\Delta X_k, \Delta y_k, \Delta Z_k)$ is just an "inexact" Newton direction for the perturbed KKT system (2.8). On the right hand side of (3.1), R_k^d, R_k^p and R_k^c are the residual components for infeasibilities and complementarity, whereas the vectors r_k^d, r_k^p, r_k^c are the residual components for the inexactness in the computed search direction.

Let $\{\sigma_k\}_{k=1}^{\infty}$ be a given sequence in $(0,1]$ such that $\bar{\sigma} := \sum_{k=0}^{\infty} \sigma_k < \infty$. We require the residual components in the inexactness in (3.1) to satisfy the following accuracy conditions:

$$\|\mathbf{A}^{+} r_k^p\| \le \gamma_p \rho \theta_k \sigma_k, \quad \|r_k^d\| \le \gamma_d \rho \theta_k \sigma_k, \quad \|r_k^c\| \le 0.5(1-\eta_2)\gamma \nu_k \mu_0. \tag{3.4}$$

Remark. In practice, we can solve (3.1) by the following procedure:

1. Compute Δy_k and ΔX_k from the following augmented system:

$$\begin{bmatrix} -Q - F_k^{-1}E_k & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{svec}\Delta X_k \\ \Delta y_k \end{bmatrix} = \begin{bmatrix} -\eta_1(\mathbf{svec}R_k^d + r_k^d) - F_k^{-1}\mathbf{svec}R_k^c \\ -\eta_1(R_k^p + r_k^p) \end{bmatrix} \tag{3.5a}$$

   with the residual vectors r_k^d and r_k^p satisfying the conditions in (3.4).

2. Compute ΔZ_k from

$$\mathbf{svec}\Delta Z_k = -F_k^{-1}E_k \mathbf{svec}\Delta X_k + F_k^{-1}\mathbf{svec}R_k^c. \tag{3.5b}$$

Here, we can see that ΔZ_k is obtained directly from (3.3) with $r_k^c = 0$. Thus, r_k^c can be ignored in the system (3.1). The dimension of the augmented system (3.5a) is $n^2 + m$, which is typically a large number even for $n = 100$. The computational cost and memory requirement for solving (3.5a) by a direct solver is about $O((n^2+m)^3)$ and $O((n^2+m)^2)$ respectively, which are prohibitively expensive for large scale problems. An iterative solver would not require the storage or manipulation of the full coefficient matrix. However, the disadvantage of using an iterative solver is the demand of good preconditioners to accelerate its convergence. In practice, constructing cheap and effective preconditioners could be the most challenging task in the implementation of an inexact interior-point algorithm for solving QSDP; see [16] for details.

After computing the search direction in (3.1), we consider the following trial iterate to determine the new iterate:

$$(\theta_k(\alpha), \nu_k(\alpha), X_k(\alpha), y_k(\alpha), Z_k(\alpha)) \tag{3.6}$$

$$= \quad ((1-\alpha\eta_1)\theta_k, (1-\alpha\eta_2)\nu_k, X_k + \alpha\Delta X_k, y_k + \alpha\Delta y_k, Z_k + \alpha\Delta Z_k), \quad \alpha \in [0,1].$$

To find the new iterate, we need to choose an appropriate step length α_k to keep the new iterate in $\mathcal{N}$. The precise choice of α_k will be discussed shortly. Before that, we present our inexact primal-dual infeasible path-following algorithm.

Algorithm IPC. Let $\theta_0 = \nu_0 = 1$. Choose parameters $\eta_1, \eta_2 \in (0,1]$ with $\eta_1 \geq \eta_2$, $\gamma_p, \gamma_d \in (0,1)$ such that $\gamma_p \leq \gamma_d$ and $\gamma_d + L\gamma_p < 1$. Pick a sequence $\{\sigma_k\}_{k=1}^{\infty}$ in $(0,1]$ such that $\bar{\sigma} := \sum_{k=0}^{\infty} \sigma_k < \infty$. Choose (X_0, y_0, Z_0) satisfying (2.2), (2.3), (2.4). Note that $(\theta_0, \nu_0, X_0, y_0, Z_0) \in \mathcal{N}$.

For $k = 0, 1, \ldots$

1. Terminate when $\nu_k < \epsilon$.
2. Find an inexact search direction $(\Delta X_k, \Delta y_k, \Delta Z_k)$ from the linear system (3.1).
3. Let $\alpha_k \in [0,1]$ be chosen appropriately so that

$$(\theta_{k+1}, \nu_{k+1}, X_{k+1}, y_{k+1}, Z_{k+1}) := (\theta_k(\alpha_k), \nu_k(\alpha_k), X_k(\alpha_k), y_k(\alpha_k), Z_k(\alpha_k)) \in \mathcal{N}.$$

Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ be the step lengths that have already been determined in the previous k iterations. For reasons that will become apparent shortly, we assume that the step lengths α_i, $i = 0, \ldots, k-1$, are contained in the interval

$$\mathcal{I} := \big[0, \min\{1, 1/(\eta_1(1+\bar{\sigma}))\}\big]. \tag{3.7}$$

Let the primal and dual infeasibilities associated with $(\theta_k(\alpha), \nu_k(\alpha), X_k(\alpha), y_k(\alpha), Z_k(\alpha))$ be

$$\begin{aligned} R_k^p(\alpha) &= \mathbf{A}(\mathbf{svec}X_k(\alpha)) - b, \\ \mathbf{svec}R_k^d(\alpha) &= -\mathbf{svec}\nabla f(X_k(\alpha)) + \mathbf{A}^T y_k(\alpha) + \mathbf{svec}Z_k(\alpha). \end{aligned}$$

We will show that $R_k^p(\alpha)$ and $R_k^d(\alpha)$ satisfy the first two conditions in $\mathcal{N}$ when α is restricted to be in the interval $\mathcal{I}$ given in (3.7).

Lemma 3.1. *Suppose the step lengths α_i associated with the iterates $(\theta_i, \nu_i, X_i, y_i, Z_i)$ are restricted to be in the interval $\mathcal{I}$ for $i = 0, \ldots, k-1$. Then we have*

$$R_k^p(\alpha) = \theta_k(\alpha)(R_0^p + \xi_k^p(\alpha)) \tag{3.8}$$

$$R_k^d(\alpha) = \theta_k(\alpha)(\mathbf{svec}R_0^d + \xi_k^d(\alpha)) \tag{3.9}$$

where

$$\|\mathbf{A}^+\xi_k^p(\alpha)\| \leq \gamma_p\rho, \quad \|\xi_k^d(\alpha)\| \leq \gamma_d\rho, \quad \forall\, \alpha \in \mathcal{I}.$$

Proof. Note that $R_k^p(\alpha)$ has exactly the same form as in the inexact interior-point algorithm considered in [19] for a linear SDP. Using the result in [19], we have

$$R_k^p(\alpha) = \theta_k(\alpha)(R_0^p + \xi_k^p(\alpha)),$$

where

$$\xi_k^p(\alpha) = \xi_k^p - \frac{\alpha\eta_1}{(1-\alpha\eta_1)\theta_k} r_k^p \;=\; -\sum_{i=0}^{k-1} \frac{\alpha_i\eta_1}{(1-\alpha_i\eta_1)\theta_i} r_i^p - \frac{\alpha\eta_1}{(1-\alpha\eta_1)\theta_k} r_k^p. \qquad (3.10)$$

The quantity $R_k^d(\alpha)$ is different from its counterpart in a linear SDP as it contains an extra term coming from the quadratic term in the objective function. Thus, we need to investigate the details. Given that the current iterate belongs to $\mathcal{N}$, we have

$$\begin{aligned}
\mathbf{svec} R_k^d(\alpha) \;&=\; -\mathbf{svec}\nabla f(X_k(\alpha)) + \mathbf{A}^T y_k(\alpha) + \mathbf{svec} Z_k(\alpha) \\
&= -\mathbf{svec}\nabla f(X_k) + \mathbf{A}^T y_k + \mathbf{svec} Z_k + \alpha[-Q(\mathbf{svec}\Delta X_k) + \mathbf{A}^T \Delta y_k + \mathbf{svec}\Delta Z_k] \\
&= \mathbf{svec} R_k^d - \alpha\eta_1(\mathbf{svec} R_k^d + r_k^d) \\
&= (1-\alpha\eta_1)\theta_k(\mathbf{svec} R_0^d + \xi_k^d) - \alpha\eta_1 r_k^d \\
&= (1-\alpha\eta_1)\theta_k\Big(\mathbf{svec} R_0^d + \xi_k^d - \frac{\alpha\eta_1}{(1-\alpha\eta_1)\theta_k} r_k^d\Big) \\
&= \theta(\alpha)(\mathbf{svec} R_0^d + \xi_k^d(\alpha)),
\end{aligned}$$

where

$$\xi_k^d(\alpha) \;=\; \xi_k^d - \frac{\alpha\eta_1}{(1-\alpha\eta_1)\theta_k} r_k^d \;=\; -\sum_{i=0}^{k-1} \frac{\alpha_i\eta_1}{(1-\alpha_i\eta_1)\theta_i} r_i^d - \frac{\alpha\eta_1}{(1-\alpha\eta_1)\theta_k} r_k^d. \quad (3.11)$$

From (3.10) and (3.11), we see that since $\alpha_i \le \frac{1}{\eta_1(1+\bar{\sigma})}$ for $i = 1, \ldots, k-1$, we have

$$\|\mathbf{A}^+ \xi_k^p(\alpha)\| \le \gamma_p \rho, \quad \|\xi_k^d(\alpha)\| \le \gamma_d \rho, \quad \forall\, \alpha \in \mathcal{I}.$$

□

Let

$$\bar{\alpha}_k = \min\Big\{1, \frac{1}{\eta_1(1+\bar{\sigma})}, \frac{0.5(1-\eta_2)\gamma\nu_k\mu_0}{\|H_{P_k}(\Delta X_k \Delta Z_k)\|}\Big\}. \qquad (3.12)$$

Next, we check the last condition in $\mathcal{N}$. The following lemma generalizes the result of Lemma 4.2 in [20].

Lemma 3.2. *For $(\theta_k, \nu_k, X_k, y_k, Z_k) \in \mathcal{N}$ and $\Delta X_k, \Delta Z_k$ satisfying (3.1), we have*

(a) $$\begin{aligned} H_{P_k}(X_k(\alpha) Z_k(\alpha)) \;&=\; (1-\alpha) H_{P_k}(X_k Z_k) + \alpha(1-\eta_2)\nu_k\mu_0 I \\ &\quad + \alpha\, \mathbf{smat} r_k^c + \alpha^2 H_{P_k}(\Delta X_k \Delta Z_k) \end{aligned}$$

(b) $$(1-\gamma)\nu_k(\alpha)\mu_0 \;\le\; \lambda_i(X_k(\alpha) Z_k(\alpha)) \;\le\; (1+\gamma)\nu_k(\alpha)\mu_0 \quad \forall\, \alpha \in [0, \bar{\alpha}_k].$$

Proof. (a) The proof of part (a) is quite standard and uses equation (3.3).
(b) The proof uses the fact that for any matrix $B \in \mathbb{R}^{n\times n}$, the real part of its spectrum is contained in the interval given by $[\lambda_{\min}(B+B^T)/2, \lambda_{\max}(B+B^T)/2]$. In particular, for any nonsingular matrix P, we have

$$\mathrm{Re}\lambda_i(B) = \mathrm{Re}\lambda_i(PBP^{-1}) \in [\lambda_{\min}(H_P(B)), \lambda_{\max}(H_P(B))] \quad \forall\, i = 1, \ldots, n.$$

Using the above fact, we have for any $i = 1, \ldots, n$,

$$\begin{aligned}
& \lambda_i(X_k(\alpha)Z_k(\alpha)) - (1-\gamma)\nu_k(\alpha)\mu_0 \\
\geq\ & \lambda_{\min}(H_{P_k}(X_k(\alpha)Z_k(\alpha))) - (1-\gamma)\nu_k(\alpha)\mu_0 \\
\geq\ & (1-\alpha)(1-\gamma)\nu_k\mu_0 + \alpha(1-\eta_2)\nu_k\mu_0 - \alpha\|r_k^c\| - \alpha^2\|H_{P_k}(\Delta X_k \Delta Z_k)\| - (1-\gamma)\nu_k(\alpha)\mu_0 \\
=\ & \alpha\gamma(1-\eta_2)\nu_k\mu_0 - \alpha\|r_k^c\| - \alpha^2\|H_{P_k}(\Delta X_k \Delta Z_k)\| \\
\geq\ & 0.5\alpha(1-\eta_2)\gamma\nu_k\mu_0 - \alpha^2\|H_{P_k}(\Delta X_k \Delta Z_k)\| \\
\geq\ & 0 \quad \text{for } \alpha \in [0\,,\,\bar{\alpha}].
\end{aligned}$$

The proof that $\lambda_i(X_k(\alpha)Z_k(\alpha)) \leq (1+\gamma)\nu_k(\alpha)\mu_0$ for all $\alpha \in [0\,,\,\bar{\alpha}]$ is similar, and we shall omit it. □

Lemma 3.3. *Under the conditions in Lemmas 3.1 and 3.2, for any $\alpha \in [0, \bar{\alpha}_k]$, we have*

$$(\theta(\alpha), \nu(\alpha), X(\alpha), y(\alpha), Z(\alpha)) \in \mathcal{N}.$$

Proof. The result follows from Lemmas 3.1 and 3.2. □

Lemma 3.4. *Suppose the conditions in (2.2), (2.3) and (2.4) hold. Then*

$$\|H_{P_k}(\Delta X_k \Delta Z_k)\| = \frac{O(1)}{(1-(\gamma_d + L\gamma_p))^2} n^2 \nu_k \mu_0. \tag{3.13}$$

The proof of Lemma 3.4 is non-trivial and we devote the next section to its proof.

We are now ready to present the main result of this paper, the polynomial iteration complexity of **Algorithm IPC**.

Theorem 1. *Let $\epsilon > 0$ be a given tolerance. Suppose the conditions in (2.2), (2.3) and (2.4) hold. Then $\nu_k \leq \epsilon$ for $k = O(n^2 \ln(1/\epsilon))$.*

Proof. From (3.12), Lemma 3.3 and Lemma 3.4, we know that

$$\alpha_i \geq \bar{\alpha} := \min\left\{1, \frac{1}{\eta_1(1+\bar{\sigma})}, \frac{O(1)}{n^2}\right\}, \quad i = 0, \ldots, k.$$

Then we have

$$\nu_k = \prod_{i=0}^{k-1}(1-\alpha_i\eta_2) \le (1-\bar{\alpha}\eta_2)^k \le \varepsilon \text{ for } k = O(n^2\ln(1/\varepsilon)).$$

□

4 Proof of Lemma 3.4

For a given $(\theta_k, \nu_k, X_k, y_k, Z_k) \in \mathcal{N}$, the purpose of Lemma 3.4 is to establish an upper bound for $\|H_{P_k}(\Delta X_k \Delta Z_k)\|$. Throughout this section, we shall consider only the NT direction, where $P_k = W_k^{-1/2}$, with $W_k \in \mathcal{S}^n_{++}$ satisfying $W_k Z_k W_k = X_k$.

It is easy to verify that

$$W_k = P_k^{-2} = Z_k^{-1/2}(Z_k^{1/2}X_kZ_k^{1/2})^{1/2}Z_k^{-1/2} = X_k^{1/2}(X_k^{1/2}Z_kX_k^{1/2})^{-1/2}X_k^{1/2}, \tag{4.1}$$

and consequently

$$\lambda_{\max}(W_k) \le \lambda_{\max}\Big((X_k^{1/2}Z_kX_k^{1/2})^{-1/2}\Big)\lambda_{\max}(X_k), \tag{4.2}$$

$$\lambda_{\min}(W_k) \ge \lambda_{\min}\Big((Z_k^{1/2}X_kZ_k^{1/2})^{1/2}\Big)\lambda_{\min}(Z_k^{-1}). \tag{4.3}$$

To facilitate our analysis, we introduce the following notation:

$$\begin{aligned}
&\widehat{X}_k = P_kX_kP_k, \quad \widehat{Z}_k = P_k^{-1}Z_kP_k^{-1};\\
&\Delta\widehat{X}_k = P_k\Delta X_kP_k, \quad \Delta\widehat{Z}_k = P_k^{-1}\Delta Z_kP_k^{-1};\\
&\widehat{E}_k = E_k(P_k^{-1} \circledast P_k^{-1}) = \widehat{Z}_k \circledast I,\\
&\widehat{F}_k = F_k(P_k \circledast P_k) = \widehat{X}_k \circledast I.
\end{aligned}$$

From the fact that $W_k^{1/2}Z_kW_k^{1/2} = W_k^{-1/2}X_kW_k^{-1/2}$, we have

$$\widehat{Z}_k = \widehat{X}_k, \quad \widehat{E}_k = \widehat{F}_k. \tag{4.4}$$

It is readily shown that $\widehat{F}_k, \widehat{E}_k, \widehat{F}_k\widehat{E}_k \in S^{\bar{n}}_{++}$. Let the eigenvalue decompositions of $\widehat{X}_k$ and $\widehat{Z}_k$ be:

$$\widehat{X}_k = \widehat{Z}_k = Q_k\Lambda_kQ_k^T, \tag{4.5}$$

where $Q_k^TQ_k = I$, $\Lambda_k = \mathrm{diag}(\lambda_k^1, \ldots, \lambda_k^n)$, and $\lambda_k^1 \le \ldots \le \lambda_k^n$. From (2.9), we have

$$(1-\gamma)\nu_k\mu_0 \le (\lambda_k^1)^2 \le \cdots \le (\lambda_k^n)^2 \le (1+\gamma)\nu_k\mu_0. \tag{4.6}$$

Let

$$\widehat{S}_k := \widehat{F}_k \widehat{E}_k^T = \frac{1}{2}(\widehat{X}_k \circledast \widehat{Z}_k + \widehat{X}_k \widehat{Z}_k \circledast I) \tag{4.7}$$
$$= \frac{1}{2}(Q_k \circledast Q_k)(\Lambda_k \circledast \Lambda_k + \Lambda_k^2 \circledast I)(Q_k \circledast Q_k)^T.$$

Then the eigenvalues of $\widehat{S}_k$ are given by

$$\Lambda(\widehat{S}_k) = \left\{ \frac{1}{4}(\lambda_i^k + \lambda_j^k)^2 : 1 \le i \le j,\ j = 1, \ldots, n \right\}.$$

From (4.5) and (4.6), we have,

$$(1-\gamma)\nu_k\mu_0 I \preceq \widehat{S}_k \preceq (1+\gamma)\nu_k\mu_0 I, \tag{4.8}$$

and

$$\|\widehat{S}_k\|_2 \le (1+\gamma)\nu_k\mu_0, \quad \|\widehat{S}_k^{-1}\|_2 \le \frac{1}{(1-\gamma)\nu_k\mu_0}. \tag{4.9}$$

Now we state a few lemmas, which lead to the proof of Lemma 3.4.

Lemma 4.1. *For any $M \in \mathbb{R}^{n\times n}$,*

$$\|(P_k \circledast P_k)\mathbf{svec}M\|^2 \le \frac{1}{(1-\gamma)\nu_k\mu_0}\|Z_k\|^2\|M\|^2,$$
$$\|(P_k^{-1} \circledast P_k^{-1})\mathbf{svec}M\|^2 \le \frac{1}{(1-\gamma)\nu_k\mu_0}\|X_k\|^2\|M\|^2.$$

Proof. First we note that $Z_k^{1/2}X_kZ_k^{1/2}$, $X_k^{1/2}Z_kX_k^{1/2}$, and X_kZ_k are similar, and $\lambda_{\min}(X_kZ_k) \ge (1-\gamma)\nu_k\mu_0$. From (4.2), (4.3), we have

$$\lambda_{\max}(W_k) \le \frac{\|X_k\|}{\sqrt{(1-\gamma)\nu_k\mu_0}}, \quad \lambda_{\min}(W_k) \ge \frac{\sqrt{(1-\gamma)\nu_k\mu_0}}{\|Z_k\|}. \tag{4.10}$$

By (4.10), we have

$$\|(P_k \circledast P_k)\mathbf{svec}M\|^2 \le \|P_k \circledast P_k\|_2^2\|M\|^2$$
$$\le \lambda_{\max}^2(W_k^{-1})\|M\|^2 \le \frac{1}{(1-\gamma)\nu_k\mu_0}\|Z_k\|^2\|M\|^2.$$

Similarly, by (4.10), we have

$$\|(P_k^{-1} \circledast P_k^{-1})\mathbf{svec}M\|^2 \le \|P_k^{-1} \circledast P_k^{-1}\|_2^2\|M\|^2$$
$$\le \lambda_{\max}^2(W_k)\|M\|^2 \le \frac{1}{(1-\gamma)\nu_k\mu_0}\|X_k\|^2\|M\|^2.$$

□

Lemma 4.2.

$$\|\mathbf{svec}\Delta\widehat{X}_k\|^2 + \|\mathbf{svec}\Delta\widehat{Z}_k\|^2 + 2\Delta\widehat{X}_k \bullet \Delta\widehat{Z}_k = \|\widehat{S}_k^{-1/2}(\mathbf{svec}R_k^c + r_k^c)\|^2,$$

$$\|H_{P_k}(\Delta X_k \Delta Z_k)\| \le \frac{1}{2}\Big(\|\mathbf{svec}\Delta\widehat{X}_k\|^2 + \|\mathbf{svec}\Delta\widehat{Z}_k\|^2\Big).$$

Proof. The last equation of (3.1) can be rewritten as

$$\widehat{E}_k(\mathbf{svec}\Delta\widehat{X}_k) + \widehat{F}_k(\mathbf{svec}\Delta\widehat{Z}_k) = \mathbf{svec}R_k^c + r_k^c. \tag{4.11}$$

Multiplying (4.11) by $\widehat{S}_k^{-1/2}$ from the left, we have

$$\mathbf{svec}\Delta\widehat{X}_k + \mathbf{svec}\Delta\widehat{Z}_k = \widehat{S}_k^{-1/2}(\mathbf{svec}R_k^c + r_k^c).$$

From here, the first equation in the lemma follows.

For the second inequality, by Lemma 4.6 of [10], we have

$$\begin{aligned}
\|H_{P_k}(\Delta X_k \Delta Z_k)\| &= \frac{1}{2}\|P_k \Delta X_k \Delta Z_k P_k^{-1} + P_k^{-1}\Delta Z_k \Delta X_k P_k\| \\
&\le \|P_k \Delta X_k \Delta Z_k P_k^{-1}\| = \|\Delta\widehat{X}_k \Delta\widehat{Z}_k\| \le \|\Delta\widehat{X}_k\|\|\Delta\widehat{Z}_k\| \\
&\le \frac{1}{2}\Big(\|\mathbf{svec}\Delta\widehat{X}_k\|^2 + \|\mathbf{svec}\Delta\widehat{Z}_k\|^2\Big).
\end{aligned}$$

□

Lemma 4.3. *We have*

$$\|\widehat{S}_k^{-1/2}(\mathbf{svec}R_k^c + r_k^c)\|^2 = O(n\nu_k\mu_0).$$

Proof. From (3.4) and (4.9), we have

$$\|\widehat{S}_k^{-1/2} r_k^c\|^2 \le \|\widehat{S}_k^{-1}\|_2 \|r_k^c\|^2 \le \frac{0.25[(1-\eta_2)\gamma\nu_k\mu_0]^2}{(1-\gamma)\nu_k\mu_0} = \frac{[(1-\eta_2)\gamma]^2\nu_k\mu_0}{4(1-\gamma)}. \tag{4.12}$$

Observe that from (4.5),

$$\mathbf{svec}R_k^c = (Q_k \circledast Q_k)\mathbf{svec}((1-\eta_2)\nu_k\mu_0 I - \Lambda_k^2).$$

Thus

$$\begin{aligned}
&\|\widehat{S}_k^{-1/2}\mathbf{svec}R_k^c\|^2 \le \|\widehat{S}_k^{-1}\|_2\|\mathbf{svec}R_k^c\|^2 \\
\le\ & \frac{1}{(1-\gamma)\nu_k\mu_0}\sum_{i=1}^n \Big((1-\eta_2)\nu_k\mu_0 - (\lambda_i^k)^2\Big)^2 \\
\le\ & \frac{n\nu_k\mu_0}{1-\gamma}(\gamma+\eta_2)^2, \quad \text{by (4.6).}
\end{aligned} \tag{4.13}$$

The required result follows from (4.12) and (4.13). This completes the proof. □

In the rest of our analysis, we introduce an auxiliary point $(\widetilde{X}_k, \widetilde{y}_k, \widetilde{Z}_k)$ whose existence is ensured by Lemma 2.1. From Lemma 3.1, we have the following equations at the kth iteration:

$$-\,\mathbf{svec}\nabla f(X_k) + \mathbf{A}^T y_k + \mathbf{svec}Z_k = \theta_k(\mathbf{svec}R_0^d + \xi_k^d),\ \|\xi_k^d\| \le \gamma_d\rho, \tag{4.14}$$

$$\mathbf{A}(\mathbf{svec}X_k) - b = \theta_k(\mathbf{svec}R_0^p + \xi_k^p),\ \|\mathbf{A}^+\xi_k^p\| \le \gamma_p\rho. \tag{4.15}$$

Thus by Lemma 2.1, there exists $(\widetilde{X}_k, \widetilde{y}_k, \widetilde{Z}_k)$ such that

$$-\,\mathbf{svec}\nabla f(\widetilde{X}_k) + \mathbf{A}^T\widetilde{y}_k + \mathbf{svec}\widetilde{Z}_k = \mathbf{svec}R_0^d + \xi_k^d \tag{4.16}$$

$$\mathbf{A}(\mathbf{svec}\widetilde{X}_k) - b = R_0^p + \xi_k^p \tag{4.17}$$

$$(1-\gamma_p)\rho I \preceq \widetilde{X}_k \preceq (1+\gamma_p)\rho I\ , \tag{4.18}$$

$$[1-(\gamma_d + L\gamma_p)]\rho I \preceq \widetilde{Z}_k \preceq [1+(\gamma_d + L\gamma_p)]\rho I. \tag{4.19}$$

Lemma 4.4. *Let*

$$\overline{X}_k = X_k - X_* - \theta_k(\widetilde{X}_k - X_*), \quad \overline{Z}_k = Z_k - Z_* - \theta_k(\widetilde{Z}_k - Z_*).$$

The following equations hold:

$$\big\langle \overline{X}_k,\, \overline{Z}_k \big\rangle = \big\langle \overline{X}_k,\, \mathcal{Q}\overline{X}_k \big\rangle, \tag{4.20}$$

$$\begin{aligned}&\big\langle \mathbf{svec}(\Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*)) + \eta_1\mathbf{A}^+ r_k^p,\, \mathbf{svec}(\Delta Z_k + \eta_1\theta_k(\widetilde{Z}_k - Z_*)) + \eta_1 r_k^d \big\rangle \\ &= \big\langle \mathbf{svec}(\Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*)) + \eta_1\mathbf{A}^+ r_k^p,\, Q\,\mathbf{svec}(\Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*))\big\rangle.\end{aligned} \tag{4.21}$$

Proof. By (4.14)–(4.17) and the fact that

$$\begin{aligned}\mathbf{A}\mathbf{svec}X_* - b &= 0,\\ -\mathbf{svec}\nabla f(X_*) + \mathbf{A}^T y_* + \mathbf{svec}Z_* &= 0,\end{aligned}$$

we have

$$\begin{aligned}\mathbf{A}\mathbf{svec}\overline{X}_k &= 0\\ \mathbf{A}^T(y_k - y_* - \theta_k(\widetilde{y}_k - y_*)) + \mathbf{svec}(\overline{Z}_k) &= Q\,\mathbf{svec}(\overline{X}_k),\end{aligned}$$

which implies (4.20). Next, by (3.1), and (4.14)–(4.17), we have

$$\begin{aligned}&\mathbf{A}\Big(\mathbf{svec}(\Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*)) + \eta_1\mathbf{A}^+ r_k^p\Big) = 0\\ &\mathbf{A}^T\Big(\Delta y_k + \eta_1\theta_k(\widetilde{y}_k - y_*)\Big) + \mathbf{svec}(\Delta Z_k + \eta_1\theta_k(\widetilde{Z}_k - Z_*)) + \eta_1 r_k^d\\ &= Q\,\mathbf{svec}(\Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*)),\end{aligned}$$

which implies (4.21). □

Let

$$T_1 = \Big(\|\mathbf{svec}\Delta\widehat{X}_k\|^2 + \|\mathbf{svec}\Delta\widehat{Z}_k\|^2\Big)^{1/2} \tag{4.22}$$

$$T_2 = \Big(\|(P_k \circledast P_k)\mathbf{svec}(\widetilde{X}_k - X_*)\|^2 + \|(P_k^{-1} \circledast P_k^{-1})\mathbf{svec}(\widetilde{Z}_k - Z_*)\|^2\Big)^{1/2} \tag{4.23}$$

$$T_3 = \Big(\|(P_k \circledast P_k)\mathbf{A}^+ r_k^p\|^2 + \|(P_k^{-1} \circledast P_k^{-1}) r_k^d\|^2\Big)^{1/2} \tag{4.24}$$

$$T_4 = \|(P_k^{-1} \circledast P_k^{-1})Q\ (\mathbf{A}^+ r_k^p)\|. \tag{4.25}$$

Then we have the following lemma.

Lemma 4.5.

$$T_1 \ \le\ 2\eta_1(\theta_k T_2 + T_3 + T_4) + \sqrt{T_5},$$

where

$$T_5 = \|\widehat{S}_k^{-1/2}(\mathbf{svec}R_k^c + r_k^c)\|^2 + 2\eta_1^2\theta_k^2\langle \widetilde{X}_k - X_*,\ \widetilde{Z}_k - Z_*\rangle + 2\eta_1^2\Big(\theta_k T_2 T_3 + T_3^2 + \theta_k T_2 T_4\Big).$$

Proof. By (4.21), we have that

$$\begin{aligned}
-\langle \Delta\widehat{X}_k,\ \Delta\widehat{Z}_k\rangle \ &=\ -\langle \Delta X_k,\ \Delta Z_k\rangle \\
=\ \ & \eta_1\theta_k[\langle \Delta X_k,\ \widetilde{Z}_k - Z_*\rangle + \langle \widetilde{X}_k - X_*,\ \Delta Z_k\rangle] + \eta_1[\langle \mathbf{svec}\Delta X_k,\ r_k^d\rangle + \langle \mathbf{A}^+ r_k^p,\ \mathbf{svec}\Delta Z_k\rangle] \\
& + \eta_1^2\theta_k[\langle \mathbf{svec}(\widetilde{X}_k - X_*),\ r_k^d\rangle + \langle \mathbf{A}^+ r_k^p,\ \mathbf{svec}(\widetilde{Z}_k - Z_*)\rangle] + \eta_1^2\langle \mathbf{A}^+ r_k^p,\ r_k^d\rangle \\
& + \eta_1^2\theta_k^2\langle \widetilde{X}_k - X_*,\ \widetilde{Z}_k - Z_*\rangle - \eta_1\langle \mathbf{A}^+ r_k^p,\ Q\ \mathbf{svec}(\Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*))\rangle \\
& - \langle \Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*),\ \mathcal{Q}\ (\Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*))\rangle.
\end{aligned}$$

Also, we have the following inequalities:

$$\begin{aligned}
&|\langle \Delta X_k,\ \widetilde{Z}_k - Z_*\rangle + \langle \widetilde{X}_k - X_*,\ \Delta Z_k\rangle| \\
&= |\langle \Delta\widehat{X}_k,\ P_k^{-1}(\widetilde{Z}_k - Z_*)P_k^{-1}\rangle + \langle P_k(\widetilde{X}_k - X_*)P_k,\ \Delta\widehat{Z}_k\rangle| \ &\le\ & T_1 T_2 \\
&|\langle \mathbf{svec}\Delta X_k,\ r_k^d\rangle + \langle \mathbf{A}^+ r_k^p,\ \mathbf{svec}\Delta Z_k\rangle| \ &\le\ & T_1 T_3 \\
&|\langle \mathbf{svec}(\widetilde{X}_k - X_*),\ r_k^d\rangle + \langle \mathbf{A}^+ r_k^p,\ \mathbf{svec}(\widetilde{Z}_k - Z_*)\rangle| \ &\le\ & T_2 T_3 \\
&|\langle \mathbf{A}^+ r_k^p,\ r_k^d\rangle| \ &\le\ & T_3^2 \\
&|\langle \mathbf{A}^+ r_k^p,\ Q\ \mathbf{svec}(\widetilde{X}_k - X_*)\rangle| \ &\le\ & T_2 T_4 \\
&|\langle \mathbf{A}^+ r_k^p,\ Q\ \mathbf{svec}\Delta X_k\rangle| \ &\le\ & T_1 T_4
\end{aligned}$$

$$-\left\langle \Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*),\, \mathcal{Q}\left(\Delta X_k + \eta_1\theta_k(\widetilde{X}_k - X_*)\right)\right\rangle \;\leq\; 0.$$

In the above, we used the Cauchy-Schwartz inequality and the fact that $ac + bd \leq \sqrt{a^2+b^2}\sqrt{c^2+d^2}$ for $a, b, c, d \geq 0$.

By Lemma 4.2, and the above inequalities, we have

$$\begin{aligned} T_1^2 &= \|\widehat{S}_k^{-1/2}(\mathbf{svec}R_k^c + r_k^c)\|^2 - 2\langle \Delta\widehat{X}_k,\, \Delta\widehat{Z}_k\rangle \\ &\leq 2\Big(\eta_1\theta_k T_1T_2 + \eta_1 T_1T_3 + \eta_1^2\theta_k T_2T_3 + \eta_1^2 T_3^2 + \eta_1^2\theta_k T_2T_4 + \eta_1 T_1T_4\Big) \\ &\quad + \|\widehat{S}_k^{-1/2}(\mathbf{svec}R_k^c + r_k^c)\|^2 + 2\eta_1^2\theta_k^2\langle \widetilde{X}_k - X_*,\, \widetilde{Z}_k - Z_*\rangle \\ &= 2\eta_1 T_1(\theta_k T_2 + T_3 + T_4) + T_5. \end{aligned}$$

The quadratic function $t^2 - 2\eta_1(\theta_k T_2 + T_3 + T_4)t - T_5$ has a unique positive root at

$$t_+ = \eta_1(\theta_k T_2 + T_3 + T_4) + \sqrt{\eta_1^2(\theta_k T_2 + T_3 + T_4)^2 + T_5},$$

and it is positive for $t > t_+$, hence we must have $T_1 \leq t_+ \leq 2\eta_1(\theta_k T_2 + T_3 + T_4) + \sqrt{T_5}$. □

Lemma 4.6. *We have*

$$T_3^2 \;=\; \frac{O(1)}{(1-(\gamma_d + L\gamma_p))^2} n^2 \nu_k \mu_0.$$

Proof. By (3.4), we have

$$\|\mathbf{A}^+ r_k^p\| \leq \theta_k\gamma_p\rho, \quad \|r_k^d\| \leq \theta_k\gamma_d\rho. \tag{4.26}$$

By Lemma 4.1 and the fact that $\|M\| \leq \mathrm{Tr}(M)$ for $M \in \mathcal{S}_+^n$, we have

$$\begin{aligned} \|(P_k \circledast P_k)\mathbf{A}^+ r_k^p\|^2 &\leq \frac{1}{(1-\gamma)\nu_k\mu_0}\|\mathbf{A}^+ r_k^p\|^2\|Z_k\|^2 \\ &\leq \frac{\gamma_p^2\rho^2}{(1-\gamma)\nu_k\mu_0}\theta_k^2\|Z_k\|^2 \leq \frac{\gamma_p^2\rho^2}{(1-\gamma)\nu_k\mu_0}\theta_k^2[\mathrm{Tr}(Z_k)]^2 \\ &= \frac{\gamma_p^2\rho^2}{(1-\gamma)\nu_k\mu_0}\frac{36}{(1-\gamma_p)^2}n^2\nu_k^2\rho^2 \;=\; \frac{O(1)}{(1-\gamma_p)^2}n^2\nu_k\mu_0 \quad \text{by Lemma 2.2.} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|(P_k^{-1} \circledast P_k^{-1})r_k^d\|^2 &\leq \frac{1}{(1-\gamma)\nu_k\mu_0}\|r_k^d\|^2\|X_k\|^2 \\ &\leq \frac{\gamma_d^2\rho^2}{(1-\gamma)\nu_k\mu_0}\theta_k^2\|X_k\|^2 \leq \frac{\gamma_d^2\rho^2}{(1-\gamma)\nu_k\mu_0}\theta_k^2[\mathrm{Tr}(X_k)]^2 \end{aligned}$$

$$= \frac{\gamma_d^2\rho^2}{(1-\gamma)\nu_k\mu_0}\frac{36}{(1-(\gamma_d+L\gamma_p))^2}n^2\nu_k^2\rho^2 = \frac{O(1)}{(1-(\gamma_d+L\gamma_p))^2}n^2\nu_k\mu_0 \quad \text{by Lemma 2.2.}$$

From here, the required result follows. □

Lemma 4.7. *Under the conditions (2.2), (2.3) and (2.4),*

$$\big\langle \widetilde{X}_k - X_*,\, \widetilde{Z}_k - Z_* \big\rangle \;\le\; 4n\mu_0.$$

Proof. The result follows from Lemma 11 in [19], and (4.18) and (4.19). □

Lemma 4.8. *Under the conditions (2.2), (2.3), and (2.4),*

$$\theta_k^2 T_2^2 \;=\; O(n^2\nu_k\mu_0).$$

Proof. By the fact that $0 \preceq \widetilde{X}_k - X_* \preceq (1+\gamma_p)\rho I$, we have

$$\begin{aligned}
&\|(P_k \circledast P_k)\mathbf{svec}(\widetilde{X}_k - X_*)\| = \|P_k(\widetilde{X}_k - X_*)P_k\| \\
\le\;& \mathrm{Tr}(P_k(\widetilde{X}_k - X_*)P_k) = \big\langle W_k^{-1},\, \widetilde{X}_k - X_* \big\rangle \\
=\;& \big\langle (Z_k^{1/2}X_kZ_k^{1/2})^{-1/2},\, Z_k^{1/2}(\widetilde{X}_k - X_*)Z_k^{1/2} \big\rangle \quad \text{by (4.1)} \\
\le\;& \lambda_{\max}((Z_k^{1/2}X_kZ_k^{1/2})^{-1/2})\big\langle Z_k,\, \widetilde{X}_k - X_* \big\rangle \\
\le\;& \frac{1}{\sqrt{(1-\gamma)\nu_k\mu_0}}\big\langle Z_k,\, \widetilde{X}_k - X_* \big\rangle.
\end{aligned}$$

Similarly, from $0 \preceq \widetilde{Z}_k - Z_* \preceq (1+\gamma_d+L\gamma_p)\rho I$, we have

$$\begin{aligned}
&\|(P_k^{-1} \circledast P_k^{-1})\mathbf{svec}(\widetilde{Z}_k - Z_*)\| = \|P_k^{-1}(\widetilde{Z}_k - Z_*)P_k^{-1}\| \\
\le\;& \mathrm{Tr}(P_k^{-1}(\widetilde{Z}_k - Z_*)P_k^{-1}) = \big\langle W_k,\, \widetilde{Z}_k - Z_* \big\rangle \\
=\;& \big\langle (X_k^{1/2}Z_kX_k^{1/2})^{-1/2},\, X_k^{1/2}(\widetilde{Z}_k - Z_*)X_k^{1/2} \big\rangle \quad \text{by (4.1)} \\
\le\;& \lambda_{\max}((X_k^{1/2}Z_kX_k^{1/2})^{-1/2})\big\langle X_k,\, \widetilde{Z}_k - Z_* \big\rangle \\
\le\;& \frac{1}{\sqrt{(1-\gamma)\nu_k\mu_0}}\big\langle X_k,\, \widetilde{Z}_k - Z_* \big\rangle.
\end{aligned}$$

Therefore, we have

$$\theta_k^2 T_2^2 \;\le\; \theta_k^2\Big(\|(P_k \circledast P_k)\mathbf{svec}(\widetilde{X}_k - X_*)\| + \|(P_k^{-1} \circledast P_k^{-1})\mathbf{svec}(\widetilde{Z}_k - Z_*)\|\Big)^2$$

$$\leq \frac{\theta_k^2}{(1-\gamma)\nu_k\mu_0}\Big(\big\langle Z_k,\, \widetilde{X}_k - X_*\big\rangle + \big\langle X_k,\, \widetilde{Z}_k - Z_*\big\rangle\Big)^2.$$

From (4.20) and the facts that $X_* \bullet Z_* = 0$, $X_k \bullet Z_*$, $X_* \bullet Z_k$, $\widetilde{X}_k \bullet Z_*$, $\widetilde{Z}_k \bullet X_* \succeq 0$, we have

$$\begin{aligned}
& \theta_k\big\langle \widetilde{X}_k - X_*,\, Z_k\big\rangle + \theta_k\big\langle X_k,\, \widetilde{Z}_k - Z_*\big\rangle \\
= \;& X_k \bullet Z_k - X_k \bullet Z_* - X_* \bullet Z_k + X_* \bullet Z_* \\
& + \theta_k\big(\big\langle X_*,\, \widetilde{Z}_k - Z_*\big\rangle + \big\langle \widetilde{X}_k - X_*,\, Z_*\big\rangle\big) + \theta_k^2\big\langle \widetilde{X}_k - X_*,\, \widetilde{Z}_k - Z_*\big\rangle \\
& - \big\langle X_k - X_* - \theta_k(\widetilde{X}_k - X_*),\, \mathcal{Q}(X_k - X_* - \theta_k(\widetilde{X}_k - X_*))\big\rangle \\
\leq \;& X_k \bullet Z_k + \theta_k\big(X_* \bullet \widetilde{Z}_k + \widetilde{X}_k \bullet Z_*\big) + \theta_k^2 \widetilde{X}_k \bullet \widetilde{Z}_k \\
\leq \;& (1+\gamma)\nu_k\mu_0 n + \theta_k(1+\gamma_d+L\gamma_p)\rho(X_* \bullet I + I \bullet Z_*) + \theta_k^2(1+\gamma_p)(1+\gamma_d+L\gamma_p)\rho^2 n \\
\leq \;& 8\nu_k\mu_0 n.
\end{aligned}$$

Thus $\theta_k^2 T_2^2 = O(n^2\nu_k\mu_0)$. □

Lemma 4.9.

$$T_4^2 = \frac{O(1)}{(1-(\gamma_d+L\gamma_p))^2} n^2\nu_k\mu_0.$$

Proof. By Lemma 4.1, we have

$$\begin{aligned}
T_4^2 &\leq \frac{1}{(1-\gamma)\nu_k\mu_0}\|X_k\|^2\|Q(\mathbf{A}^+ r_k^p)\|^2 \\
&\leq \frac{1}{(1-\gamma)\nu_k\mu_0}\|X_k\|^2 L^2\|\mathbf{A}^+ r_k^p\|^2 \\
&\leq \frac{\gamma_p^2\rho^2 L^2}{(1-\gamma)\nu_k\mu_0}\theta_k^2\|X_k\|^2 \\
&\leq \frac{\gamma_p^2 L^2}{(1-\gamma)\nu_k}\frac{O(1)}{(1-(\gamma_d+L\gamma_p))^2} n^2\nu_k^2\rho^2, \quad \text{by Lemma 2.2} \\
&= \frac{O(1)}{(1-(\gamma_d+L\gamma_p))^2} n^2\nu_k\mu_0.
\end{aligned}$$

□

The following proof directly leads to Lemma 3.4.

Lemma 4.10.

$$T_1^2 = \frac{O(1)}{(1-(\gamma_d+L\gamma_p))^2} n^2\nu_k\mu_0.$$

Proof. From Lemma 4.5 to Lemma 4.9 and the fact that $(a+b)^2 \leq 2a^2+2b^2$, we have

$$\begin{aligned} T_1^2 &\leq \left(2\eta_1(\theta_k T_2 + T_3 + T_4) + \sqrt{T_5}\right)^2 \\ &\leq 8(\theta_k T_2 + T_3 + T_4)^2 + 2T_5 \\ &\leq 8(\theta_k T_2 + T_3 + T_4)^2 + 2\|\widehat{S}_k^{-1/2}(\mathbf{svec}R_k^c + r_k^c)\|^2 + 4\theta_k^2 \langle \widetilde{X}_k - X_*, \, \widetilde{Z}_k - Z_* \rangle \\ &\quad +4\theta_k T_2 T_3 + 4T_3^2 + 4\theta_k T_2 T_4 \\ &\leq \frac{O(1)}{(1-(\gamma_d + L\gamma_p))^2} n^2 \nu_k \mu_0 + O(n\nu_k\mu_0). \end{aligned}$$

□

Thus, by Lemma 4.2 and Lemma 4.10, we have

$$\|H_{P_k}(\Delta X_k \Delta Z_k)\| \leq \frac{1}{2}T_1^2 = \frac{O(1)}{(1-(\gamma_d + L\gamma_p))^2} n^2 \nu_k \mu_0.$$

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