Filtering Random Flows with Noisy Lagrangian Tracers

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Joint work with Andrew J, Majda and Nan Chen
CAOS-CIMS

Thursday 4th December, 2014
Information Barriers for Noisy Lagrangian Tracers in Filtering Random Incompressible Flows, Nonlinearity 27 (2014) 2133-2163;

Consider a signal process $U_s$ and observation process $X_s$:

\[
\begin{align*}
    dU_s &= b(Z_s)ds + \Sigma dW_s, \\
    dX_s &= h(Z_s)ds + \sigma dB_s, \\
    Z_s &= (U_s, X_s).
\end{align*}
\]

Kushner’s equation describes the evolution of $\mathbb{E}_t f(U_t)$:

\[
d\mathbb{E}_t f_t = \mathbb{E}_t \mathcal{L} f_t dt + \langle \mathbb{E}_t (f_t h_t) - \mathbb{E}_t f_t \mathbb{E}_t h_t, (\sigma \sigma^T)^{-1} (dX_t - \mathbb{E}_t h_t dt) \rangle
\]

with $f_t = f(U_t)$, $h_t = h(Z_t)$ and $\Sigma \Sigma^T = (a_{i,j})$:

\[
\mathcal{L} f = \sum b_i \partial u_i f + \frac{1}{2} \sum a_{i,j} \partial^2_{u_i, u_j} f.
\]

A closure type of problem $\mathbb{E}_t f_t$ requires $\mathbb{E}_t (f_t h_t)$.

Not practical to solve at all.
Suppose the dynamics is linear conditioned on $X_s$:

$$dU_s = [B(X_s)U_s + b(X_s)]ds + \Sigma dW_s,$$

$$dX_s = [H(X_s)U_s + h(X_s)]ds + \sigma dB_s.$$ 

The conditional distribution $P(U_t \in \cdot | X_{s \leq t})$ is given by a Gaussian distribution $\mathcal{N}(m_t, R_t)$:

$$\dot{m}_t = [B(X_t)m_t + b(X_t)] + R_t H(X_t)^T (\sigma \sigma^T)^{-1} [\dot{X}_t - B(X_t)m_t - b(X_t)],$$

$$\dot{R}_t = B(X_t)R_t + R_t B^T(X_t) + \Sigma \Sigma^T - R_t H(X_t)^T (\sigma \sigma^T)^{-1} H(X_t) R_t.$$ 

Assuming the initial distribution is a Gaussian.
Jan 1992, some containers were washed into ocean;
28,800 ducks were released;
Travel with the currents;
Oceanographer Curtis Ebbesmeyer.
Lagrangian Tracers: Oceanography

Media source: http://www.argo.ucsd.edu/
Practical Need

- Wide range of application: lab experiments, oceanography, atmospheric science, environmental science;
- Heavy loads of data assimilation: how to recover the underlying velocity field;

![Graphs of True velocity field, Observations, and Posterior mean velocity field]
Lagrangian Data Assimilation

- How do we model the underlying dynamics?
- How do we filter the underlying velocity field using the Lagrangian tracers?
- A. Apte, CKRT. Jones, AM. Stuart, ...
- Nonlinear filters, EnKF, particle filters.

Filter Performance

- How does the filter behave as the number of tracers, $L$, increases?
- How do we evaluate the information gain? Optimize against $L$?
- Are there simplified filters based on geophysical intuition?
Theoretical Questions

Lagrangian Data Assimilation

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Filter Performance

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- How do we evaluate the information gain? Optimize against $L$?
- Are there simplified filters based on geophysical intuition?
Framework: conditional Gaussian theory
Non-dimensional linearized shallow water equation:

\[
\frac{\partial \vec{v}}{\partial t} + \varepsilon^{-1} \vec{v}^\perp = -\varepsilon^{-1} \nabla h, \quad \frac{\partial h}{\partial t} + \varepsilon^{-1} \nabla \cdot \vec{v} = 0.
\]

\(\varepsilon = U/Lf\) is the Rossby number;

General Solution:

\[
\begin{bmatrix}
\vec{v}(\vec{x}, t) \\
h(\vec{x}, t)
\end{bmatrix} = \sum_{\vec{k} \in \mathbb{Z}^2, \alpha \in \{0, \pm\}} \hat{v}_{\vec{k},\alpha} \exp(i\vec{k} \cdot \vec{x} - i\omega_{\vec{k},\alpha} t) \vec{r}_{\vec{k},\alpha}.
\]

Geostrophic balanced (GB) modes + gravity modes:

\[
\begin{aligned}
\text{GB} & \quad \begin{cases}
\vec{r}_{\vec{k},0} \perp \vec{k} \\
\omega_{\vec{k},0} = 0
\end{cases} & \quad \text{gravity} & \quad \begin{cases}
\vec{r}_{\vec{k},\pm} \not\perp \vec{k} \\
\omega_{\vec{k},\pm} = \varepsilon^{-1} \sqrt{\delta |\vec{k}|^2 + 1}
\end{cases}
\end{aligned}
\]
Setup: random incompressible flow

We model a dimensional two velocity field as superposition of Fourier modes of random amplitude:

\[ \vec{v}(\vec{x}, s) = \sum_{\vec{k} \in K, \alpha} \hat{v}_{\vec{k}, \alpha}(s) \exp(i\vec{k} \cdot \vec{x}) \vec{r}_{\vec{k}, \alpha} \]

- \( \hat{v}_{\vec{k}, \alpha} \) are independent O.U. processes

\[ d\hat{v}_{\vec{k}, \alpha}(s) = (-\gamma_{\vec{k}} + i\omega_{\vec{k}, \alpha})\hat{v}_{\vec{k}, \alpha} \, ds + f_{\vec{k}, \alpha} \, ds + \sigma_{\vec{k}, \alpha} \, dW_{\vec{k}, \alpha}(s). \]

Require \( \hat{v}_{\vec{k}, \alpha} = \hat{v}^*_{-\vec{k}, -\alpha} \) to ensures real valued and divergence free.

- Stochastic version of many deterministic linearized models, cf. Majda & Harlim.

Filtering: recover the signals \( \hat{v}_{\vec{k}, \alpha} \).
There are $L$ tracers inside the velocity field, each of them follows:

$$d\vec{X}_l(s) = \vec{v}(\vec{X}_l(s), s)ds + \sigma_x dW_{l,x}(s)$$

Tracers are also perturbed by random turbulence or instrumental noise, this is modeled by $\sigma_x dW_{l,x}(s)$. It can be seen as observation error in filtering.

Conditioned on the value of $\vec{v}_{s \leq t}$, $\vec{X}_1(s), \ldots, \vec{X}_L(s)$ are independent processes;

This conditional distribution is written as $P_{\vec{v}_{s \leq t}}$;

The distribution that includes all possible realization of $\vec{v}_{s \leq t}$ will be called the “super-ensemble” probability $P$. 
Practical Importance of Conditional Probability

Result in

- Super Ensemble Probability, $\mathbf{P}$: averaged over all weather realization;
- Distribution conditioned on each possible weather, $\mathbf{P}_{\vec{v}_s \leq t}$ for a.s. $\vec{v}_s \leq t$: holds for all weather realization.
Whole signal vector: \( \mathbf{U}_s = (, \ldots, \hat{v}_{\vec{k},\alpha}(s), \ldots)^T \). Its dynamics can be written jointly as

\[
d\mathbf{U}_s = -\Gamma \mathbf{U}_s ds + \vec{F}_s ds + \Sigma_v dW_s
\]

Assume \( \Gamma > 0 \), \( \vec{F}_s \) is periodic, then \( \mathbf{U}_s \) has a equilibrium distribution, which is \( \pi_{att} = \mathcal{N}(m_{att}^t, R_{att}) \). This is the information without the filtering.

Velocity field formulation:

\[
\vec{v}(\vec{x}, s) = \sum_{\vec{k} \in K, \alpha} \hat{v}_{\vec{k},\alpha}(s) \exp(i\vec{k} \cdot \vec{x})\vec{r}_{\vec{k},\alpha} = P_X(\vec{x})\mathbf{U}_s
\]

\( P_X(\vec{x}) = [\ldots, \exp(i\vec{k} \cdot \vec{x}), \ldots] \) is periodic with period \([0, 2\pi]^2\).
Whole signal vector: $U_s = (\ldots, \hat{v}_{\vec{k},\alpha}(s), \ldots)^T$. Its dynamics can be written jointly as

$$dU_s = -\Gamma U_s ds + \vec{F}_s ds + \Sigma_v dW_s$$

Assume $\Gamma > 0$, $\vec{F}_s$ is periodic, then $U_s$ has a equilibrium distribution, which is $\pi_{att} = \mathcal{N}(m^a_{att}, R_{att})$. This is the information without the filtering.

Velocity field formulation:

$$\vec{v}(\vec{x}, s) = \sum_{\vec{k} \in K, \alpha} \hat{v}_{\vec{k},\alpha}(s) \exp(i\vec{k} \cdot \vec{x}) \vec{r}_{\vec{k},\alpha} = P_X(\vec{x}) U_s$$

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Whole signal vector: \( \mathbf{U}_s = (\ldots, \hat{v}_{\vec{k},\alpha}(s), \ldots)^T \). Its dynamics can be written jointly as

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Filtering: Observations

- Rewrite the evolution of tracers as:

\[ d\vec{X}_l(s) = \vec{v}(\vec{X}_l(s), s)ds + \sigma_x dW_{l,x}(s) \]

\[ = P_X(\vec{X}_l(s))U_s ds + \sigma_x dW_{l,x}(s) \]

- Nonlinear in \( \vec{X}_l(s) \) but linear in \( U_s \).

- Regroup the observations together: \( X_s = (\vec{X}_1(s), \ldots, \vec{X}_L(s))^T \),

\[ dX_s = P_X(X_s)U_s ds + \sigma_x dW_x(s) \]

\[
\begin{bmatrix}
\vec{X}_1(s) \\
\vdots \\
\vec{X}_L(s)
\end{bmatrix}
= 
\begin{bmatrix}
P_X(\vec{X}_1(s)) \\
\vdots \\
P_X(\vec{X}_L(s))
\end{bmatrix}
U_s ds + \sigma_x d
\begin{bmatrix}
W_{1,x}(s) \\
\vdots \\
W_{L,x}(s)
\end{bmatrix}
\]
Rewrite the evolution of tracers as:

\[
d\vec{X}_l(s) = \vec{v}(\vec{X}_l(s), s)ds + \sigma_x dW_{l,x}(s) = P_X(\vec{X}_l(s)) U_s ds + \sigma_x dW_{l,x}(s)
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- Nonlinear in \(\vec{X}_l(s)\) but linear in \(U_s\).
- Regroup the observations together: \(X_s = (\vec{X}_1(s), \ldots, \vec{X}_L(s))^T\),

\[
d\begin{pmatrix} \vec{X}_1(s) \\ \vdots \\ \vec{X}_L(s) \end{pmatrix} = \begin{pmatrix} P_X(\vec{X}_1(s)) \\ \vdots \\ P_X(\vec{X}_L(s)) \end{pmatrix} U_s ds + \sigma_x d\begin{pmatrix} W_{1,x}(s) \\ \vdots \\ W_{L,x}(s) \end{pmatrix}
\]
Jointly, the signal $v_s$ and $X_s$ follow:

$$dU_s = -\Gamma U_s ds + \vec{F}_s ds + \Sigma_v dW_v(s)$$

$$dX_s = P_X(X_s)U_s ds + \sigma_x dW_x(s)$$

Filtering: find the distribution of $U_s$ given the observation of $X_{s \leq t}$.

When $P_X(X_s)$ is a deterministic matrix, this will be a Kalman-Bucy filter.
Jointly, the signal $v_s$ and $X_s$ follow:

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- Conditional Gaussian: the conditional distribution of $U_t \sim \pi_{t|t}$ is $\mathcal{N}(m_t, R_t)$, where $m_0 = m_0^{att}$, $R_0 = R_{att}$, and

$$dm_t = -\Gamma m_t dt + \vec{F}_t dt + \sigma_x^{-2} R_t P^*_X(X_t)(dX_t - P_X(X_t)m_t dt)$$

$$dR_t = [-\Gamma R_t - R_t \Gamma^* + \Sigma_v \Sigma^*_v - R_t P^*_X(X_t)P_X(X_t)R_t] dt$$

- Both optimal and concrete!
Simplifications make the filtering theory here clean and viable:

- Periodic domain and no topography;
- Linear flow dynamics;
- Continuous time observation;
- Direct observation of noisy Lagrangian tracers, against noisy observation of exact tracers:

\[ dx_t = \bar{v}(x_t)dt, \quad dY_t = x_t dt + \sigma_x dB_t. \]
The additional information of $v_t$ from the observation of the tracers $\{X_l(s)\}_{s \leq t, l \leq L}$ is measured by the difference between $\pi_{t|t} = \mathcal{N}(m_t, R_t)$ and climatological state $\pi_{att} = \mathcal{N}(m_{t, att}, R_{att})$.

This can be measured by the relative entropy, which is defined:

$$\mathcal{P}(p, q) := \int p(x) \log \frac{p(x)}{q(x)} dx$$

When $p \sim \mathcal{N}(m_p, R_p)$, $q \sim \mathcal{N}(m_q, R_q)$, there is an explicit formula

$$\mathcal{P}(p, q) = \left( \frac{1}{2} (m_p - m_q)^* R_q^{-1} (m_p - m_q) \right) \text{ Signal} + \left( \frac{1}{2} \log \left( \frac{\det R_q}{\det R_p} \right) - \frac{N}{2} + \frac{1}{2} \text{tr}(R_p R_q^{-1}) \right) \text{ Dispersion}$$
The additional information of $v_t$ from the observation of the tracers $\{X_l(s)\}_{s \leq t, l \leq L}$ is measured by the difference between $\pi_{t|t} = \mathcal{N}(m_t, R_t)$ and climatological state $\pi_{att} = \mathcal{N}(m_{t}^{att}, R_{att})$; this can be measured by the relative entropy, which is defined:

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$$\mathcal{P}(p, q) = \left( \frac{1}{2} (m_p - m_q)^* R_q^{-1} (m_p - m_q) \right) \quad \text{Signal}$$

$$+ \left( \frac{1}{2} \log \left( \frac{\det R_q}{\det R_p} \right) - \frac{N}{2} + \frac{1}{2} \text{tr} \left( R_p R_q^{-1} \right) \right) \quad \text{Dispersion}$$
Apply it to $\pi_{t|t} = \mathcal{N}(m_t, R_t)$ and $\pi_{att} = \mathcal{N}(m^{att}_t, R_{att})$:

$$
P(\pi_{t|t}, \pi_{att}) = \left( \frac{1}{2} (m_t - m^{att}_t) R^{-1}_{att} (m_t - m^{att}_t) \right)$$

$$+ \left( \frac{1}{2} \log \left( \frac{\det R_{att}}{\det R_t} \right) - \frac{N}{2} + \frac{1}{2} \text{tr}(R_t R^{-1}_{att}) \right)$$

We can get a quantitative analysis of each parts as long as we can analyze $R_t$ and $m_t$;

Especially interested to see how does $R_t$ and $m_t$ change as $L$ increases.
Incompressible: Explicit Asymptotic Features
Focus on geostrophic balanced (GB) modes first:

\[ \vec{v}^B(\vec{x}, s) = \sum_{\vec{k} \in K, \alpha = 0} \hat{v}_{\vec{k}, \alpha}(s) \exp(i \vec{k} \cdot \vec{x}) \vec{r}_{\vec{k}, \alpha}. \]

Incompressible: \( \vec{r}_{\vec{k}, 0} \perp \vec{k} \Rightarrow (\vec{v}^B)^\perp = -\nabla h; \)
Denote the corresponding tracers as
\( \vec{X}^B_l(s) : d\vec{X}^B_l(s) = \vec{v}^B(\vec{X}^B_l(s), s)ds + \sigma_x dW_l(s). \)
Represents the large scale geophysical flow. Figure source: ECMWF
Theorem

Conditioned on a.s. realization of $\vec{v}_{s \geq 0}^B$, the distribution of the noisy tracers $\vec{X}^B_i(s)$ converges geometrically fast towards the uniform distribution on $[0, 2\pi]^2$.

- Invariance: Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} [\vec{v}_i^B(\vec{x}, t)p(\vec{x}, t)] + \frac{\sigma^2_x}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} [\delta_{ij}p(\vec{x}, t)].$$

- Convergence: a coupling argument based on the Girsanov theorem.
Assume: tracers are initially deployed independently and uniformly.

Media source: http://www.argo.ucsd.edu/
Recovery of streamlines

Truth

Lagrangian Tracer Filtering

Xin Tong
Theorem (Posterior Covariance Part)

For any fixed $t > 0$, and almost surely $\vec{v}_{s \leq t}$, $R_t$ converges to $R_L$ as $L \to \infty$:

$$\sqrt{L}\|R_t - R_L\| \to 0 \quad \text{in} \quad \mathbb{P} \vec{v}_{s \leq t}.$$ 

$R_L$ is a diagonal matrix with entries of form $\frac{\sigma_{k,0}^2}{\gamma + \sqrt{\gamma^2 + L\sigma_x^2\sigma_{k,0}^2|\vec{r}_{k,0}^2|}}$. 

From this, we can tell both $\|R_L\|$ and $\|R_t\|$ are of order $L^{-\frac{1}{2}}$. 
Recall the evolution of the covariance:

\[
dR_t = [-\Gamma R_t - R_t \Gamma^* + \Sigma_v \Sigma_v^* - \sigma_x^{-2} R_t P_X^*(X_t) P_X(X_t) R_t] dt
\]

Recall that

\[
P_X(X_t) = \begin{bmatrix}
\ldots & \exp(i k \vec{X}_1^B(t)) \vec{r}_k & \ldots \\
\ldots & \exp(i k \vec{X}_2^B(t)) \vec{r}_k & \ldots \\
\ldots & \ldots & \ldots & \ldots 
\end{bmatrix}_{N \times 2L}
\]

The quadratic term is a sum of conditionally i.i.d. random matrices, \([P_X^*(X_t) P_X(X_t)]_{j,k} = \sum_{l=1}^L \exp(i(\vec{k} - \vec{j}) \cdot \vec{X}_l^B(t))(\vec{r}_j \vec{r}_k)\).

Law of large numbers:

\[
P_X^*(X_t) P_X(X_t) \approx L \mathbb{E}^\pi \exp(i(\vec{k} - \vec{j}) \cdot \vec{X}_l^B(t))(\vec{r}_j \vec{r}_k)]_{j,k} = LI_{2L}.
\]

The limit \(R_L\) is the attractor of the mean field equation:

\[
dR_t = [-\Gamma R_t - R_t \Gamma^* + \Sigma_v \Sigma_v^* - \sigma_x^{-2} LR_t^2] dt.
\]
Posterior covariance:
Rescaled Deviation: $\sqrt{L} \| R_L - R_t \|$:
Theorem (Posterior Mean Part)

There is a fixed time $s_0 > 0$ so that for any fixed $t > s_0$, and almost surely $\tilde{v}_{s \leq t}$:

$$|m_t - U_t| \rightarrow 0 \quad \text{in} \quad P_{\tilde{v}_{s \leq t}}$$

- This means the posterior mean $m_t$ can capture the underlying signal $U_t$.
- Quick way to see this: $E_t(m_t - U_t) \otimes (m_t - U_t) = R_t$. 
- Posterior Mean in mode $[1, 1]$,
Recall that:

\[
P(\pi_t|t,\pi_{att}) = \left( \frac{1}{2} (m_t - m_{att})^* R_{att}^{-1} (m_t - m_{att}) \right) \quad \text{Signal}
\]

\[
+ \left( \frac{1}{2} \log \left( \frac{\det R_{att}}{\det R_t} \right) - \frac{N}{2} + \frac{1}{2} \text{tr}(R_t R_{att}^{-1}) \right) \quad \text{Dispersion}
\]

Since \( \sqrt{L\|R_L - R_t\|} \to 0 \) and \( |m_t - U_t| \to 0 \), each part should approach the limit where \( R_t \) replaced by \( R_L \) and \( m_t \) replaced by \( U_t \).
Theorem (Relative Entropy)

There exists a fixed time $s_0 > 0$ such that for a.s. realization of $\vec{v}_{s \leq t}$

- For any $t > s_0$,
  \[
  \text{Signal} \to \frac{1}{2} (U_t - m_t^{\text{att}})^* R_{\text{att}}^{-1}(U_t - m_t^{\text{att}}) \quad \text{in} \quad P_{\vec{v}_{s \leq t}}.
  \]

- For any $t > 0$,
  \[
  \frac{\text{Dispersion}}{\frac{N}{4} \log L} \to 1 \quad \text{in} \quad P_{\vec{v}_{s \leq t}}.
  \]

- This means the reduction of uncertainty grows logarithmly w.r.t number of tracers;

- To acquire more information, we require exponentially more tracers.
Relative Entropy:

\[ \frac{N}{4} \log 10 \approx 5.7, \ N = 25. \]
Compressible: fast-slow systems
Compressible case

- Consider the full flow: \( \vec{v}(\vec{x}, s) = \sum_{\vec{k} \in K, \alpha} \hat{v}_{\vec{k}, \alpha}(s) \exp(i\vec{k} \cdot \vec{x}) \vec{r}_{\vec{k}, \alpha} \);
- \( d\hat{v}_{\vec{k}, \alpha}(s) = (-\gamma + i\omega_{\vec{k}, \alpha})\hat{v}_{\vec{k}, \alpha} ds + f_{\vec{k}, \alpha} ds + \sigma_{\vec{k}, \alpha} dW_{\vec{k}, \alpha}(s) \);
- GB part \( \omega_{\vec{k}, 0} = 0 \) and gravity waves \( \omega_{\vec{k}, \pm} = \varepsilon^{-1} \sqrt{|\vec{k}|^2 + 1} \);
- \( \hat{v}_{\vec{k}, \pm}(s) \sim \exp(i\omega_{\vec{k}, \pm} s)\tilde{v}_{\vec{k}, \pm}(s) \), fast and slow components;
- Geostrophic approximation, fast wave average phenomena, at a fixed location \( \vec{x} \)

\[ \int_{0}^{t} \hat{v}_{\vec{k}, \pm}(s)e^{ik \cdot \vec{x}} ds \sim \varepsilon. \]
Geostrophic approximation

- Hence the geostrophically balanced flow $\vec{v}^B$ is more interesting to recover as the slow dynamics;
- Yet Lagrangian tracers’ trajectories depend nonlinearly over the fast gravity waves $d\vec{X}_l(s) = \vec{v}(\vec{X}_l(s), s)ds + \sigma_x dW^x_l(s)$:

$$\int_0^t \hat{v}_{k,\pm}(s)e^{ik\cdot\vec{X}_l(s)}ds \sim \epsilon?$$

- The features of incompressible case do not necessarily apply here.

<table>
<thead>
<tr>
<th>Signal</th>
<th>Observation</th>
<th>Estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perfect: $\vec{v}<em>t = \sum</em>{k,\alpha} \vec{v}_t$</td>
<td>$\rightarrow X_l(t)$</td>
<td>$\rightarrow \pi_{t</td>
</tr>
<tr>
<td>$\text{proj}$</td>
<td>$\ll$</td>
<td>$\text{proj} \ll$</td>
</tr>
<tr>
<td>GB-Theory: $\vec{v}<em>t^B = \sum</em>{k,0} \vec{v}_t^B$</td>
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Comparison between two cases

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<td>$\vec{v}<em>t = \sum</em>{k,\alpha} \vec{X}<em>l(t) \overset{\text{proj}}{\Rightarrow} \pi</em>{t</td>
<td>t} = \mathcal{N}(m_t, R_t)$</td>
<td>$\vec{v}^B_t = \sum_{k,0} \vec{X}^B_l(t) \overset{\text{proj}}{\Rightarrow} \pi^B_{t</td>
</tr>
</tbody>
</table>

Theorem

Assume both filters are started from the same Gaussian distribution with no correlation between GB and gravity parts, then for any fixed $q \geq 1, T \geq 0$ there is an $M = M(q, T, \vec{v}^B_{s \geq 0})$,

$$[\mathbb{E}_{\vec{v}^B_{s \geq 0}} \sup_{t \leq T} \| R^B_t - \mathbf{P} R_t \|^{2q}]^{\frac{1}{2q}} \leq \varepsilon M, \quad [\mathbb{E}_{\vec{v}^B_{s \geq 0}} \sup_{t \leq T} \| m^B_t - \mathbf{P} m_t \|^q]^{\frac{1}{q}} \leq \varepsilon M,$$

for a.s. realization of $\vec{v}^B_{s \geq 0}$. 
Simplified sub-optimal filtering strategy

- Reverse thinking: ignoring the gravity waves is an ignorable model error?
- A simplified filter based only on GB parts:
  - Reduced dimensionality;
  - Can be run at a slow system pace;
  - Fast systems may require unclear information.

\[
\vec{v}_t = \sum_{k,\alpha} \vec{X}_l(t)_{\text{sub-opt}} \pi_{t|t}(X_l(s \leq t)) = \mathcal{N}(m_t^r, R_t^r)
\]

\[
\vec{v}_t = \sum_{k,\alpha} \vec{X}_l(t)_{\text{opt}} \pi_{t|t} = \mathcal{N}(m_t, R_t)
\]

\[
\vec{v}_t^B = \sum_{k,0} \vec{X}_l^B(t)_{\text{opt}} \pi_{t|t}(X_l^B(s \leq t))
\]
Simplified sub-optimal filtering strategy

- Reverse thinking: ignoring the gravity waves is an ignorable model error?
- A simplified filter based only on GB parts:
  - Reduced dimensionality;
  - Can be run at a slow system pace;
  - Fast systems may require unclear information.

\[ \vec{v}_t = \sum_{\vec{k},\alpha} X_l(t) \xrightarrow{\text{sub-opt}} \pi_{t|t}^B(X_l(s \leq t)) = \mathcal{N}(m_t^r, R_t^r) \]

\[ \vec{v}_t = \sum_{\vec{k},\alpha} X_l(t) \xrightarrow{\text{opt}} \pi_{t|t} = \mathcal{N}(m_t, R_t) \]

\[ \vec{v}_t^B = \sum_{\vec{k},0} X_l^B(t) \xrightarrow{\text{opt}} \pi_{t|t}^B(X_l^B(s \leq t)) \]
Reduced: \[
\vec{v}_t = \sum_{k,\alpha} \vec{k}_{\alpha} \rightarrow X_l(t) \rightarrow \pi^B_{t|t}(X_l(s \leq t)) = \mathcal{N}(m^r_t, R^r_t)
\]

Perfect: \[
\vec{v}_t = \sum_{k,\alpha} \vec{k}_{\alpha} \rightarrow X_l(t) \rightarrow \pi_{t|t} = \mathcal{N}(m_t, R_t)
\]

**Theorem**

Assume both filters are started from the same Gaussian distribution with no correlation between GB and gravity parts, then for any fixed \( q \geq 1, T \geq 0 \) there is an \( M = M(q, T) \),

\[
\mathbb{E} \sup_{t \leq T} \| R^r_t - P R_t \|^{2q} \frac{1}{2q} \leq \varepsilon M, \quad \mathbb{E} \sup_{t \leq T} \| m^r_t - P m_t \|^{q} \frac{1}{q} \leq \varepsilon M.
\]
- Recovered Streamlines with different $L$:

$\varepsilon = 0.1$:

$\varepsilon = 1$:

- Key difference between two regimes: “compressibility” of the tracers.
Trajectories of tracers with or without gravity waves:

- \( \varepsilon = 0.1, \sigma_x = 0.1 \)
- \( \varepsilon = 0.1, \sigma_x = 0.5 \)
- \( \varepsilon = 1, \sigma_x = 0.1 \)
- \( \varepsilon = 1, \sigma_x = 0.5 \)
Posterior Mean with different $L$:

$\varepsilon = 0.1$

$\varepsilon = 1$
RMS error with different $L$:

GB mode

Gravity mode

$\varepsilon = 0.1$

$\varepsilon = 1$
Uncertainty reduction with different $L$:

\[ \epsilon = 0.1 \]

\[ \epsilon = 1 \]
Key difference between compressible and incompressible flow;
The reduced filter works well when $\varepsilon \ll 1$;
The statistics of the GB filter $\pi^B_{t|t}$ collapses with the one of $\pi_{t|t}$ when $\varepsilon \ll 1$. 
A filter for Lagrangian data, both exact and explicit;
Concrete asymptotic features for incompressible flows;
Such features also apply to compressible flows when $\varepsilon \ll 1$;
A simplified filter works well when $\varepsilon \ll 1$. 
Thank You