Geometric ergodicity for Stochastic Lattice Models

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Outline

1. MJO $\Rightarrow$ Stochastic Lattice $\Rightarrow$ PDMP.
2. Geometric ergodicity under Wasserstein distance.
3. An abstract theorem and application.
MJO: dominant mode of tropical intraseasonal variability.

Media source: youtube and NCAR
Failure of GCM

Key features:

1. Slow eastward phase speed 5 m/s.
2. Dispersion: $d\omega/dt \approx 0$.
3. Horizontal quadruple structure.
4. Intermittent MJO events.
5. Wave trains: generation and demise.

from Lin et al. (2006)
Skeleton Models

- **Key features:**
  1. Slow eastward phase speed 5m/s.
  2. Dispersion: \( d\omega/dt \approx 0 \).
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  5. Wave trains: generation and demise.

- **Skeleton model:**

\[
\begin{align*}
\partial_t u - yv - \partial_x \theta &= 0, \\
yu &= \partial_y \theta, \\
\partial_t \theta - (\partial_x u + \partial_y v) &= \bar{H}a - s^\theta \\
\partial_t q + \bar{Q}(\partial_x u + \partial_y v) &= -\bar{H}a + s^q, \\
[\partial_t a &\approx \Gamma qa].
\end{align*}
\]

\( u \): zonal \( v \): meridional \( \theta \): temp \( q \): moisture, \( a \): synoptic activity.
Truncations on the $y, z$ direction, discretization on the $x$ direction

$(u, v, q, \theta) \Rightarrow (K_i, R_i, Z_i)_{i \leq N}$ follows an ODE,

$a \Rightarrow (A_i)_{i \leq N}$ is a birth/death process.

\[
\begin{align*}
\dot{K}_i + D_x^+ K_i &= (s_i^\theta - \bar{H} A_i)/2 - \bar{d} K_i, \\
\dot{R}_i - D_x^- R_i/3 &= (s_i^\theta - \bar{H} A_i)/3 - \bar{d} R_i, \\
\dot{Z}_i &= (s_i^\theta - \bar{H} A_i)(1 - \bar{Q}) - \bar{d} Z_i,
\end{align*}
\]

\[
\frac{d}{dh} \mathbb{P}(A_i(t + h) = A_i + 1 | \mathcal{F}_t) = \gamma Q_i^+ A_i + 1_{A_i=0},
\]

\[
\frac{d}{dh} \mathbb{P}(A_i(t + h) = A_i - 1 | \mathcal{F}_t) = \gamma Q_i^- A_i,
\]

\[Q_i := Z_i - \bar{Q}(K_i + R_i)\]
Transformation formulas

Continuous version:

\[ u = \cos z [(K - R)\phi_0 + R\phi_2 / \sqrt{2}] , \quad v = 4 \cos z(\partial_x R + S^\theta - \bar{H} A)\phi_1 / 3\sqrt{2} , \]
\[ \theta = - \sin z [(K + R)\phi_0 + R\phi_2 / \sqrt{2}] , \quad q = Q\phi_0 \sin z , \]
\[ p = \cos z [(K + R)\phi_0 + R\phi_2 / \sqrt{2}] , \quad w = (\partial_x u + \partial_y v) \sin z , \]

with Hermite functions:

\[ \phi_0 = \sqrt{2}(4\pi)^{-1/4} \exp(-y^2 / 2) , \]
\[ \phi_1 = 2y(4\pi)^{-1/4} \exp(-y^2 / 2) , \]
\[ \phi_2 = (2y^2 - 1)(4\pi)^{-1/4} \exp(-y^2 / 2) . \]
Dynamical systems based on lattice structures, consist of

- ODE systems $X_t = (X_{i,t})_{i \in I}$ with neighboring interaction representing partial derivatives.
- A Markov jump process $Y_t = (Y_{i,t})_{i \in I}$ on a countable space.

Simplest tropical climate model, $\eta_i$ represents the CIN,

$$\begin{align*}
\frac{dK_i}{dt} + D_x^+ K_i &= -\frac{\bar{d} + d_\theta + d_{sh}}{2} K_i - \frac{d_\theta + d_{sh} - \bar{d}}{2} R_i - (d_\theta \theta_{eq,i} + d_{sh} \theta_{s,i} + P_i), \\
\frac{dR_i}{dt} - D_x^- R_i &= -\frac{\bar{d} + d_\theta + d_{sh}}{2} R_i - \frac{d_\theta + d_{sh} - \bar{d}}{2} K_i - (d_\theta \theta_{eq,i} + d_{sh} \theta_{s,i} + P_i), \\
\frac{dZ_i}{dt} &= -dq Z_i + \frac{d_\theta + d_{sh} - dq}{2} \bar{Q}(K_i + R_i) + \bar{Q}(d_\theta \theta_{eq,i} + d_{sh} \theta_{s,i}) + dq q_{s,i} - (1 - \bar{Q}) P_i, \\
P_i &= \frac{l - \eta_i}{\tau cl} \left( Z_i + \frac{\alpha + \bar{Q}}{2} (K_i + R_i) - \bar{q} \right)^+, \\
\frac{d}{dh} \mathbb{P}(\eta_i(t + h) = \eta_i + 1|\mathcal{F}_t) &= \frac{l - \eta_i}{\tau I}, \\
\frac{d}{dh} \mathbb{P}(\eta_i(t + h) = \eta_i - 1|\mathcal{F}_t) &= \frac{\eta_i}{\tau I} \exp \left( -2U_0 \frac{\eta_i - 1}{l - 1} + \bar{\gamma}q_i - h_0 \right).
\end{align*}$$
Mathematical setup

- Piecewise deterministic Markov process (PDMP): an ODE $X_t \in \mathbb{R}^d$ coupled with a Markov jump process $Y_t$ in a countable state space $F$:
  \[
  \begin{cases}
  dX_t = -b(X_t, Y_t)dt \\
  \mathbb{P}(Y_{t+h} = \tilde{y}|Z_t) = \lambda(Z_t, \tilde{y})h + o(h)
  \end{cases}
  \]

  Notation: $Z_t = (X_t, Y_t)$ and $z = (x, y)$. $\bar{\lambda}(z) := \sum_{\tilde{y} \neq y} \lambda(z, \tilde{y})$.

- Piecewise deterministic: given jump times $t_i$ and jump sites $y_i$
  \[X_t = \Psi^{y_n}_{t-t_n} \circ \Psi^{y_{n-1}}_{t_n-t_{n-1}} \circ \cdots \circ \Psi^{y_0}_{t_1} X_0.\]

- Ideal models for intermittent phenomenon: different dynamical regimes between jump times.
Possible simplifications:

- **Constant rate:** $\lambda(x, y, y') = \lambda(y, y')$, $Y_t$ is a Markov process by itself.
- **Bounded rates:** the total rate $\bar{\lambda}(z) = \sum_{y'} \lambda(z, y')$ is bounded from above (and away from zero).
- **Piecewise contraction:** flow generated by each regime is contracting $|\Psi^y_t x - \Psi^y_t x'| \leq e^{-\gamma t} |x - x'|$, e.g.

  \[
  \langle b(x, y) - b(x', y), x - x' \rangle \leq -\gamma |x - x'|^2.
  \]

Stochastic lattice models: non-constant rates, unbounded rates, piecewise contracting

\[
\begin{align*}
\dot{K}_i + D_x^+ K_i &= (s_i^\theta - \tilde{H} A_i)/2 - \tilde{d} K_i, \\
\dot{R}_i - D_x^- R_i/3 &= (s_i^\theta - \tilde{H} A_i)/3 - \tilde{d} R_i, \\
\dot{Z}_i &= (s_i^\theta - \tilde{H} A_i)(1 - \bar{Q}) - \tilde{d} Z_i, \\
Q_i := Z_i - \bar{Q}(K_i + R_i)
\end{align*}
\]

\[
\begin{align*}
c_a &= \gamma Q_i^+ A_i + 1_{A_i=0}, \\
c_d &= \gamma Q_i^- A_i,
\end{align*}
\]
Possible simplifications:

- **Constant rate:** $\lambda(x, y, y') = \lambda(y, y')$, $Y_t$ is a Markov process by itself.

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  $$\langle b(x, y) - b(x', y), x - x' \rangle \leq -\gamma |x - x'|^2.$$ 

Stochastic lattice models: non-constant rates, unbounded rates, piecewise contracting

$$\begin{align*}
\dot{K}_i + D^+_x K_i &= \left(s^\theta_i - \bar{H} A_i\right)/2 - \bar{d} K_i, \\
\dot{R}_i - D^-_x R_i/3 &= \left(s^\theta_i - \bar{H} A_i\right)/3 - \bar{d} R_i, \\
\dot{Z}_i &= (s^\theta_i - \bar{H} A_i)(1 - \bar{Q}) - \bar{d} Z_i, \\
Q_i &:= Z_i - \bar{Q}(K_i + R_i)
\end{align*}$$

\begin{align*}
c_a &= \gamma Q_i^+ A_i + 1_{A_i = 0}, \\
c_d &= \gamma Q_i^- A_i.
\end{align*}$$
A stochastic process $Z_t$ is geometrically ergodic, if

$$d(P_t^* \mu, P_t^* \nu) \leq e^{-\gamma t} C_{\mu,\nu}.$$ 

Indicates uniqueness of invariant measure $\pi$, and

$$d(P_t^* \mu, \pi) \leq e^{-\gamma t} C_{\mu,\pi}.$$ 

Important consequence: Birkhoff ergodic theorem:

$$\frac{1}{t} \int_0^t f(Z_s) ds \xrightarrow{t \to \infty} \int f(z) \pi(dz).$$
Wasserstein distance

Wasserstein-1 distance generated by $d$:

$$d(\mu, \nu) = \inf \int d(z, z') C_{\mu, \nu}(dz, dz')$$

Coupling: $C_{\mu, \nu}$ has marginal distributions $\mu$ and $\nu$.

$$d(P^*_t \mu, P^*_t \nu) = \inf C^{\mu, \nu} d(Z_t, Z'_t).$$
Discrete distance, $d(z, z') = 1_{z \neq z'}$, leads to total variation distance.

$$d(\mu, \nu) = \int |\mu(x) - \nu(x)| dx.$$ 

Geometric distance, e.g. $d(x, x') = |x - x'|$.

Our distance:

$$d(z, z') = \sqrt{|x - x'|^2 + 1_{y \neq y'}}.$$ 

Euclidean distance is better for contracting process, e.g.

$$dX_t = -X_t dt.$$
Classical total variation distance (Meyn, Tweedie, Mattingly, Stuart)

- Find a Lyapunov function $V$ with compact sublevel sets,

\[ \mathbb{E}V(Z_t) \leq Ce^{-\gamma t}V(Z_0) + K. \]

Absorbing ball property: recurrence to $\{V \leq M\}$.

- Reachability: two processes from two points in $\{V \leq M\}$, can get to the same point with probability bounded from below.
Framework for proving geometric ergodicity

General Wasserstein distance (Hairer, Mattingly, Stuart)

- Find a Lyapunov function $V$ with compact sublevel sets,

$$\mathbb{E} V(Z_t) \leq C e^{-\gamma t} V(Z_0) + K.$$  

Absorbing ball property: recurrence to $\{V \leq M\}$.

- Reachability: two processes from two points in $\{V \leq M\}$, can get close with probability bounded from below.

- Asymptotic coupling: points close enough can get closer:

$$d(P_{t_0}^* \delta_z, P_{t_0}^* \delta_{z'}) < \frac{1}{2} d(z, z') \quad \text{if} \quad d(z, z') < \epsilon.$$
Theorem (Majda and T.)

For piecewise contracting processes, it suffices to check the following:

- There is a Lyapunov function $V(x, y)$ that dominates $|x|^2, \bar{\lambda}(x, y)$ and $\partial_x \lambda(x, y, y')$.

- There is a regime $y_c$ that is reachable from other points, i.e. $P^z(Y_t = y_c) > 0$ for a time $t$.

Then there is geometric ergodicity under the Wasserstein distance generated by

$$d(z, z') = \sqrt{|x - x'|^2 + \mathbb{1}_{y \neq y'}}.$$
Lyapunov function can be obtained from notions of energy

\[ E = \sum_{i} \frac{1}{2} \left[ 2K_i^2 + 3R_i^2 + \frac{(Z_i + 1)^2}{(1 - Q)Q} \right] + \frac{\bar{H}\Delta A\eta_i}{\Gamma Q} + 1. \]

\[
\begin{cases}
\dot{K}_i + D^+_{x} K_i = \left( s_i^\theta - \bar{H} A_i \right)/2 - \bar{d}K_i, \\
\dot{R}_i - D^-_{x} R_i/3 = \left( s_i^\theta - \bar{H} A_i \right)/3 - \bar{d}R_i, \\
\dot{Z}_i = \left( s_i^\theta - \bar{H} A_i \right)(1 - Q) - \bar{d}Z_i, \\
Q_i := Z_i - \bar{Q}(K_i + R_i)
\end{cases}
\]

dissipative energy

\[ \mathcal{L}E = \lim_{t \to 0} \frac{1}{t} [\mathbb{E}E_t - \mathbb{E}E_0] \leq -\gamma E + \sum k_v(s_i^\theta) \]

\[ \mathbb{E}E_t \leq e^{-\gamma t}E_0 + \gamma^{-1}\left[ \sum k_v(s_i^\theta) \right]. \]

In order to dominate all the terms, use \( E^3 \) instead.
Nontrivial exercise to show $A_i \equiv 1$ is reachable.

\[
\begin{align*}
\dot{K}_i + D_x^+ K_i &= (s_i^\theta - \bar{H} A_i)/2 - \bar{d} K_i, \\
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Q_i &= Z_i - \bar{Q}(K_i + R_i)
\end{align*}
\]

In fact, could stuck if $s_i^\theta = \bar{H} A_i$, which is uninteresting.

$\sum s_i^\theta$ not being a multiple of $\bar{H} A_i$ avoids it.

Basic idea:

- For each fixed $y = (A_i)$, investigate its corresponding attractor.
- At attractor we can change some values of $A_i$ to 1.
Theorem (Majda and T.)

For the stochastic skeleton MJO model with $\sum s_i^\theta$ not being a multiple of $\bar{H}A_i$, the model is geometrically ergodic under the distance

$$d(z, z') = \sqrt{|x - x'|^2 + 1_{y \neq y'}}.$$  

A similar theorem for the simplest tropical model also exist, but requires additional condition to obtain a Lyapunov function.
Stochastic lattice models as a tool to capture intermittent features.

- They are PDMP with unbounded rates, and in many cases piecewise contracting.
- They are geometrically ergodic under a Wasserstein distance, if there is a dominating Lyapunov function and reachability of one state.
The restrictive piecewise contraction condition can be replaced by a "contraction on average" condition.

Simulation theory of the stochastic lattice model:

- Convergence in finite time:
  \[ d(P_T^*\mu, \hat{P}_T^*\mu) = O(\sqrt{\Delta}). \]

- Convergence to the equilibrium measure:
  \[ d(\pi, \hat{P}_{nT}^*\mu) \to 0. \]
Geometric Ergodicity for Piecewise Contracting Processes with Applications for Tropical Stochastic Lattice Models, accepted by CPAM, Feb 2015.
Links and slides can be found at www.cims.nyu.edu/~tong.

Thank you!