A PDE for the multi-time joint probability of the Airy process

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1. Introduction

The Airy process can be defined as the limit of the Dyson Brownian motion, as we are going to do later. However, it also appears in various statistical physical models, such as the polynuclear growth process [1,2] and the Domino tiling model [3]. Since the Airy process is stationary with continuous sample paths [1], we can pick any time \( t \) and consider the gap probability that all particles are in \((-\infty, u)\), denoted by \( P(u) \), and find that the probability is given by the GUE Tracy–Widom distribution [4]

\[
P(u) = e^{-\int_{-\infty}^{u}(s-u)q(s)ds},
\]

where \( q(s) \) is the solution of the Painlevé II equation

\[
q''(s) = sq(s) + 2q^2(s), \quad q(s) \approx \begin{cases} e^{-(2/3)s^{3/2}} & \text{for } s \to \infty, \\ 2\sqrt{-s/2} & \text{for } s \to -\infty. \end{cases}
\]

In their study of the joint gap probability for several times of the Airy process, Prähofer and Spohn [1] posed the problem to find a PDE for the joint gap probability. Adler and van Moerbeke [5] solved the problem for the 2-time case, and assuming a plausible conjecture of the boundary condition, got the asymptotic expansion of the probability function \( P(t, u, v) \), which is the probability that all particles are in \((-\infty, u)\) initially and in \((-\infty, v)\) after a time \( t \). Their solution was obtained by a previous result of theirs on the spectrum of coupled random matrices [6]. They regarded the joint distribution for the Dyson Brownian motion of 2-time as a \( \tau \) function of the two-Toda lattice, and construct a PDE with variables in times and boundary points of the Dyson Brownian motion as a consequence of identities for \( \tau \) functions and Virasoro identities specific to the situation. Then they got the PDE for the Airy process by taking the limit.

This paper generalizes their result to the multi-time case, and the technical heart is the same identity for \( \tau \) functions, although in the generalized case we need more elaborate work to fit differential operators in times and boundary points of the Dyson Brownian motion into the structure of two-Toda \( \tau \) functions.

After the description of the problem, we state the PDEs for both the Dyson Brownian motion with finite number of particles and its limit, the Airy process with infinitely many particles, and an example for the 3-time \((m = 2)\) case for the Airy process. Section 2 derives the result for the Dyson process and Section 3 derives the result for the Airy process by taking a limit.

1.1. Description of the model

The free Brownian motion process is determined by the transition probability distribution

\[
P(t, \tilde{X}, X) = \frac{1}{\sqrt{2\pi t/\beta}} e^{-\frac{(X-\tilde{X})^2}{2t/\beta}},
\]

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The definition of the Airy process in this paper is slightly different from the definition in these papers. See Remark 2.
where $\tilde{X}$ and $X$ are initial and terminal coordinates of the particle, and $\beta$ is the diffusion constant. The probability distribution $P(t, \tilde{X}, X)$, as a function of $t$ and $X$, satisfies the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{1}{2\beta} \frac{\partial^2}{\partial X^2} P.$$ 

If we add a harmonic potential $\rho X^2/2$ to the process, then the probability distribution $P(t, \tilde{X}, X)$ satisfies (see e.g. [7])

$$\frac{\partial P}{\partial t} = \left[ \frac{1}{2\beta} \frac{\partial^2}{\partial X^2} - \frac{\partial}{\partial X} (-\rho X) \right] P,$$

and the process is determined by ($c = e^{-\beta t}$)

$$P(t, \tilde{X}, X) = \frac{1}{\sqrt{\pi(1-c^2)}} e^{-\frac{(\beta \tilde{X} - \rho X)^2}{(1-c^2)/\rho \beta}}.$$ 

While the free Brownian motion process is dispersive, the Brownian motion process in the harmonic potential well has a stationary distribution

$$P(X) = \frac{e^{-\rho X^2}}{\sqrt{\pi / \rho \beta}}.$$ 

Now we can define the Ornstein–Uhlenbeck process [8] of an $n \times n$ Hermitian matrix $B$, in which all the $n^2$ real variables -- $n$ for real diagonal entries, $n(n-1)/2$ for the real parts of off-diagonal entries, and the other $n(n-1)/2$ for the imaginary parts of them -- are in independent Brownian motion in harmonic potential wells. The $\rho$ for them is uniformly 1, and $\beta$ is 1 for the $n$ diagonal variables and 2 for the $n(n-1)$ off-diagonal variables. Therefore for $i,j$ in $\{1, \ldots, n\}$, ($c = e^{-t}$)

$$P_{ii}(t, \vec{B}_i, B_i) = \frac{1}{\sqrt{\pi(1-c^2)}} e^{-\frac{(\beta \vec{B}_i - \rho B_i)^2}{1-c^2}};$$

$$P_{ij}(t, \vec{B}_i, \vec{B}_j) = \frac{1}{\sqrt{\pi(1-c^2)/2}} e^{-\frac{(\beta \vec{B}_i - \rho B_i)(\beta \vec{B}_j - \rho B_j)}{(1-c^2)/2}};$$

$$P_{ij}(t, \vec{B}_i, \vec{B}_j) = \frac{1}{\sqrt{\pi(1-c^2)/2}} e^{-\frac{(\beta \vec{B}_i - \rho B_i)(\beta \vec{B}_j - \rho B_j)}{(1-c^2)/2}},$$

and we can write the joint transition probability distribution as

$$P(t, \vec{B}, B) = \prod_{i=1}^{n} P_{ii}(t, B_i) \prod_{1 \leq i < j \leq n} P_{ij}(t, B_i, B_j) = \frac{C^{-1}}{(1-c^2)^{n^2/2}} e^{-\frac{\text{Tr}(\beta \vec{B} - \rho B)^2}{1-c^2}}.$$ 

We consider the multi-time transition function with the initial state $B_0$ at $t_0 = 0$, the terminal state $B_m$ and a series of intermediate states $B_1, \ldots, B_{m-1}$, such that the time between state $B_0$ and $B_i$ is $t_i$. If we denote

$$s_i = \begin{cases} 0 & i = 0, \\ t_1 & i = 1, \\ t_i - t_{i-1} & i = 2, \ldots, m, \end{cases}$$

and

$$c_i = e^{-s_i},$$

then

$$P(t_1, \ldots, t_m; B_0, \ldots, B_m) = C^{-1} \prod_{i=1}^{m} e^{-\frac{\text{Tr}(\beta \vec{B}_i - \rho B_i)^2}{1-c_i^2}}.$$ 

The Ornstein–Uhlenbeck process has a stationary distribution

$$P(B) = C^{-1} e^{-\text{Tr} B^2}. \quad (1)$$

Since the Ornstein–Uhlenbeck process is invariant under the unitary transformation, we define the process of the eigenvalues as the Dyson Brownian motion process [9], whose multi-time transition probability distribution is ($0 = t_0 < t_1 < \cdots < t_m$)

$$P(t_1, \ldots, t_m; \lambda^{(0)}, \ldots, \lambda^{(m)}) = \text{The transition probability of the } n \times n \text{ Hermitian matrix with eigenvalues initially } \lambda^{(0)} = (\lambda^{(0)}_1, \ldots, \lambda^{(0)}_m) \text{ and } \lambda^{(1)} \text{ after time } t_1, \lambda^{(2)} \text{ after time } t_2, \ldots, \text{ and } \lambda^{(m)} \text{ after the total time } t_m.$$

---

2 Through out this paper, $C$ stands for various constants, which we do not bother to write down explicitly.
If we change the coordinates of the $\mathbb{R}^{n^2}$ space of $n \times n$ Hermitian matrices in to the eigenvalue-angle coordinates $\lambda_1, \ldots, \lambda_n, \theta_1, \ldots, \theta_{n(n-1)}$, with the Jacobian identity (see e.g. [10])

$$\prod_{i=1}^{n} \prod_{1 \leq j < k \leq n} (d\theta(y_j) d\lambda(y_j)) = V(\lambda)^2 \prod_{i=1}^{n} d\lambda_i \prod_{i=1}^{n(n-1)} d\theta_i,$$

where $V(\lambda) = \prod_{1 \leq j < k \leq n} (\lambda_i - \lambda_j)$ is the Vandermonde, we find the explicit formula for $P(t_1, \ldots, t_m; \lambda^{(0)}, \ldots, \lambda^{(m)}):$

$$P(t_1, \ldots, t_m; \lambda^{(0)}, \ldots, \lambda^{(m)}) = \frac{1}{C} \int \cdots \int \prod_{i=1}^{m} e^{-\frac{1}{4} \sum_{i=1}^{n} \lambda_i^2} \prod_{i=1}^{m} V(\lambda^{(i)})^2 \prod_{i=1}^{m} \prod_{j=1}^{n} \prod_{k=1}^{m} d\theta_i,$$

where $\theta^{(i)}$ appears in the integral but is not relevant to the result, since the transition probability is independent of $\theta^{(i)}$ for the unitary invariant property.

By the Harish–Chandra–Itzykson–Zuber (HCIZ) formula [11]

$$\int_{U(n)} e^{Tr(UVU^{-1})} dU = C \det(e^{\lambda^{(j)}_i}) \frac{1}{V(\lambda) V(Y)},$$

where $X = \text{diag}(x_1, \ldots, x_n)$ and $Y = \text{diag}(y_1, \ldots, y_n)$ are diagonal matrices, we can evaluate the multi-time transition probability density as

$$P(t_1, \ldots, t_m; \lambda^{(0)}, \ldots, \lambda^{(m)}) = \frac{1}{C} V(\lambda^{(0)})^{-1} V(\lambda^{(m)}) \prod_{i=1}^{m} \det \left( e^{\frac{1}{4} \sum_{i=1}^{n} \lambda_i (\frac{1}{2} - t_i)} \right) \times e^{-\frac{1}{c_1} \sum_{i=1}^{n} \lambda_i^2} \prod_{i=1}^{m} \left( e^{-\frac{1}{4} \lambda_i^2 \sum_{j=1}^{i} \lambda_j^2} \right) e^{-\frac{1}{1-r_{det}} \sum_{i=1}^{n} \lambda_i^2}.$$

If we take the initial state with eigenvalues $\lambda^{(0)}$ from the stationary distribution (1), which is

$$\tilde{P}(\lambda^{(0)}) = \frac{1}{C} V(\lambda^{(0)})^2 e^{-\frac{1}{2} \sum_{i=1}^{n} \lambda_i^2}.$$

We get the multi-time correlation function in the stationary Dyson process

$$\tilde{P}(t_1, \ldots, t_m; \lambda^{(0)}, \ldots, \lambda^{(m)}) = \tilde{P}(\lambda^{(0)}) P(t_1, \ldots, t_m; \lambda^{(0)}, \ldots, \lambda^{(m)}) = \frac{1}{C} \prod_{i=1}^{m} \det \left( e^{\frac{1}{4} \sum_{i=1}^{n} \lambda_i (\frac{1}{2} - t_i)} \right) \times e^{-\frac{1}{c_1} \sum_{i=1}^{n} \lambda_i^2} \prod_{i=1}^{m} \left( e^{-\frac{1}{4} \lambda_i^2 \sum_{j=1}^{i} \lambda_j^2} \right) e^{-\frac{1}{1-r_{det}} \sum_{i=1}^{n} \lambda_i^2}.$$
where $\chi^j_i$ is the indicator function defined as

$$\chi^j_i(t) = \begin{cases} 0 & \text{if } t \in \bigcup_{i=1}^0 (a_{2i-1}, a_{2i}), \\ 1 & \text{otherwise}, \end{cases}$$

and ($\Lambda$ stands for the Airy function)

$$K^j_i(x, y) = \begin{cases} \int_0^\infty \Lambda_i(x + z) \Lambda_j(y + z) dz & \text{if } i = j, \\ \int_0^\infty e^{-(i-j)z} \Lambda_i(x + z) \Lambda_j(y + z) dz & \text{if } i > j, \\ \int_{-\infty}^0 e^{(i-j)z} \Lambda_i(x + z) \Lambda_j(y + z) dz & \text{if } i < j. \end{cases}$$

Then we can define the Airy process, which contains infinitely many particles, by the multi-time joint gap probability (6). Furthermore, we are going to give a PDE satisfied by $\tau$ with variables $t_1$ and $\bar{a}_i^{(l)}$.

**Remark 1.** To make the definition (6) meaningful, we need $\bar{a}_i^{(l)}$ to be $-\infty$ for all $l$. Otherwise the left-hand side of (6) is 0 and the right-hand side is not well defined.

**Remark 2.** We should emphasis a subtle difference between definitions. In this paper, we regard the Airy process as a limiting process of the $n$-particle Dyson Brownian motion as $n \to \infty$. Thus in the Airy process there are infinitely many particles. However, in the original definition of the Airy process in [1], it is defined as the process of the rightmost particle (whose existence is proved in [1]) among the infinitely many particles, and the $\infty$-particle process, which we call the Airy process, is called the ensemble of world lines of the Airy field. For details see Section 4 of [1], especially the definition 4.2. Most papers follow the definition in [1], e.g. [2,3], but in [5] our version of the definition of the Airy process is implicitly used, which is natural because in [5] the Sine process is studied in parallel, and the Sine process is another limiting process of the $n$-particle Dyson Brownian motion as $n \to \infty$, and has infinitely many particles. Although we follow the convention in [5], the PDE we get still solves the problem posed in [1,2].

1.2. Notational convenience

Throughout this paper, parentheses (...) always include numbers and functions; brackets [...] always include operators; braces {...} are always for Wronskians; $[f, g]_D = gD - fD$, where $D$ is a differential operator.

1.3. Statement of main results

With notations defined in Section 1.1, we define differential operators ($l = 0, 1, \ldots, m$)

$$D^{l-1} = \sum_{i=1}^{2l-1} \frac{\partial}{\partial \bar{a}_i^{(l)}}, \quad D^{l} = \sum_{k=1}^{2l} \frac{\partial}{\partial \bar{a}_k^{(l)}},$$

if all $\bar{a}_i^{(l)}$ are finite; otherwise we drop the $\bar{a}_1^{(l)}$ (resp. $\bar{a}_m^{(l)}$) part if $\bar{a}_1^{(l)} = -\infty$ (resp. $\bar{a}_m^{(l)} = \infty$). Then we denote

$$A_1 = \sum_{l=0}^m e^{-t} D^{l-1}, \quad B_1 = \sum_{l=0}^m e^{t} D^{l-1},$$

$$A_2 = \sum_{l=0}^m e^{-2l} D^{l} + \sum_{l=1}^m (1 - e^{-2l}) \frac{\partial}{\partial t} e^{-2l},$$

$$B_2 = \sum_{l=0}^m e^{2l} D^{l} + \sum_{l=1}^m (e^{2l} - e^{-2l}) \frac{\partial}{\partial t} e^{-2l}. \quad \text{(10)}$$

Now we state

**Theorem 1** (Dyson Brownian Motion). Given $t_1, \ldots, t_m$, the logarithm of the joint gap distribution for the stationary Dyson Brownian motion $P_n^{Dyson}$ defined in (2) (abbreviated as $P_n$) satisfies a third order non-linear PDE in times and boundary points of $U^{(l)}$

$$A_1 \frac{B_2 A_1 \log P_n}{B_1 A_1 \log P_n + 2ne^{-t_n}} = B_1 \frac{A_2 B_1 \log P_n}{A_1 B_1 \log P_n + 2ne^{-t_n}}. \quad \text{(11)}$$
Similarly with the notations
\[
D = \sum_{l=0}^{m} D^{l,1}, \\
D_{1l} = \sum_{l=0}^{m} (t_{m} - t_{l}) D^{l,1}, \\
D_{1R} = \sum_{l=0}^{m} t_{l} D^{l,1}, \\
D_{1} = D_{1l} - D_{1R} = \sum_{l=0}^{m} (t_{m} - 2t_{l}) D^{l,1}, \\
D_{2} = \sum_{l=0}^{m} ((t_{m} - t_{l})^{2} + t_{l}^{2}) D^{l,1}, \\
D_{3} = \sum_{l=0}^{m} ((t_{m} - t_{l})^{3} - t_{l}^{3}) D^{l,1}, \\
E = \sum_{l=0}^{m} D^{1,2}, \\
E_{1} = \sum_{l=0}^{m} (t_{m} - 2t_{l}) D^{1,2}, \\
\mathcal{T}_{1} = 2 \sum_{l=1}^{m} t_{l} \frac{\partial}{\partial t_{l}}, \\
\mathcal{T}_{2} = 2 \sum_{l=1}^{m} t_{l}(t_{m} - t_{l}) \frac{\partial}{\partial t_{l}},
\]
we state the result for the Airy process

**Theorem 2 (Airy Process).** Given \( t_{1}, \ldots, t_{m} \), the logarithm of the joint gap probability for the Airy process \( P^{\text{Airy}} \) defined in (6) (abbreviated as \( \log P \)) satisfies a third order non-linear PDE in times and boundary points of \( U^{(t)} \)

\[
D^{2}[\mathcal{E}_{1} + \mathcal{D}_{3} + \mathcal{T}_{2}] \log P - \mathcal{D} D_{1} \mathcal{E}_{1} + \mathcal{D}_{1} \mathcal{T}_{1} \log P - 2 \mathcal{D}_{1l} \mathcal{D}_{1R} \mathcal{D}_{1} \log P = \{ D^{2} \log P, D \mathcal{D}_{1} \log P \} D.
\]

(12)

In the case of \( m = 1 \), our results agree with those in [5]. Especially, if \( U^{(0)} = (-\infty, u), U^{(1)} = (\infty, v) \) and denote \( t_{1} = t \), then the result for \( \log P^{\text{Airy}}(t; u, v) \) is

**Corollary 1 ([5]).** The logarithm of the 2-time joint gap probability for the Airy process \( P^{\text{Airy}}(t; u, v) \) (abbreviated as \( \log P \)) satisfies a third-order non-linear PDE in variables \( u, v \) and \( t \)

\[
\left[ (v-u) \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \frac{\partial^{2}}{\partial u \partial v} + t \left[ \frac{\partial^{2}}{\partial u^{2}} - \frac{\partial^{2}}{\partial v^{2}} \right] \frac{\partial}{\partial t} + t^{2} \left[ \frac{\partial^{2}}{\partial u} - \frac{\partial}{\partial v} \right] \right] \log P = \frac{1}{2} \left\{ \frac{\partial^{2}}{\partial u^{2}} - \frac{\partial^{2}}{\partial v^{2}} \right\} \log P .
\]

(13)

In the \( m = 2 \) case, if \( U^{(0)} = (-\infty, u), U^{(1)} = (\infty, v), U^{(2)} = (\infty, u), t_{1} = t, t_{2} = s \), the result for \( \log P^{\text{Airy}}(t, s; u, v, w) \) is

**Corollary 2.** The logarithm of the 3-time joint gap probability for the Airy process \( P^{\text{Airy}}(t, s; u, v, w) \) (abbreviated as \( \log P \)) satisfies a third-order non-linear PDE in variables \( u, v, w, t \) and \( s \)

\[
\left[ t(u-v) \frac{\partial^{2}}{\partial u \partial v} + s(u-w) \frac{\partial^{2}}{\partial u \partial w} + (s-t)(v-w) \frac{\partial^{2}}{\partial v \partial w} + \left[ -s \frac{\partial}{\partial u} + (2t-s) \frac{\partial}{\partial v} + s \frac{\partial}{\partial w} \right] \left[ t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s} \right] + t(s-t) \frac{\partial}{\partial t} \right]
\]

\[
\times D \log P + \left[ -t^{3} \frac{\partial^{3}}{\partial u^{2} \partial v} - s^{3} \frac{\partial^{3}}{\partial u^{2} \partial w} + t^{3} \frac{\partial^{3}}{\partial u \partial v \partial w} + (2t-s)(2s-t) \frac{\partial^{3}}{\partial u \partial v \partial w} \right] \log P
\]

\[
+ s^{3} \frac{\partial^{3}}{\partial u \partial v^{2}} + (s-t)^{3} \frac{\partial^{3}}{\partial v^{2} \partial w} \right\} \log P
\]

\[
= \frac{1}{2} \left\{ -s \frac{\partial}{\partial u} + (2t-s) \frac{\partial}{\partial v} + s \frac{\partial}{\partial w} \right\} \log P .
\]

(14)
1.4. Relation to other results

For the 2-time joint gap probability of the Airy process, Adler and van Moerbeke not only found the PDE (13), but also computed the asymptotic expansion of \( P^{\text{Airy}}(t; u, v) \) with respect to \( t \to \infty \). Their method of computation cannot be analogously applied in an obvious way to the general multi-time situation. Even for the 3-time case, the author is unable to analyze the asymptotic behavior of log \( P^{\text{Airy}}(t; s; u, v, w) \) when \( t, s \to \infty \) by (14) and simple boundary conditions. Widom found another method for computing the asymptotic behavior of \( P^{\text{Airy}}(t; u, v) \) in [14], which is based on results in [4, 15]. It is an interesting question whether one can get such asymptotic behavior for the multi-time situation by that method.

To find equations satisfied by the gap probabilities in various matrix models, one can get PDEs by Adler and van Moerbeke’s method, or get systems of differential equations by Tracy and Widom’s method. The two kinds of results appear to be quite different, albeit they describe the same model. For example, the reader may compare the results in our paper and the corresponding results in [16]. Nevertheless, these two approaches stem from the same integrable structure: the Toda lattice. For the Gaussian unitary ensemble, which is equivalent to the Dyson Brownian motion at a single snapshot, comparisons between the two approaches have been done in [17, 18]. The exact relation between our results of the multi-time joint gap probability and those in [16] is still an open problem.

In this paper we utilize the fact that the joint gap probability of the Dyson Brownian motion is a specialization of the two-Toda \( \tau \) function. Now mounting evidence suggests that it is also an isomonodromic \( \tau \) function in Jimbo, Miwa and Ueno’s sense [19]. For the 1-time gap probability, which is the same as the gap probability in the Gaussian unitary ensemble, it has been proved in [20]. For the 2-time joint gap probability, Bertola et al. have been making steady progress, e.g. [21]. The relation between isomonodromic \( \tau \) functions and Toda \( \tau \) functions is not very clearly understood. Our PDE may help clarify that relation.

2. The joint probability in the Dyson Brownian motion

To get the PDE, we need to consider a generalized integral in which for all \( l = 1, \ldots, m, t^{(l)}_i (i = 1, 2, \ldots) \) and \( c^{(l)}_{ij} (i, j = 1, 2, \ldots) \) are formal variables,

\[
\tau_n(t^{(l)}_i, c^{(l)}_{ij}; a^{(l)}_i) = \frac{1}{C} \int \cdots \int_{U(0)^n \times \cdots \times U(m)^n} V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{l=0}^m \prod_{i=1}^n \sum_{k=1}^\infty \frac{c^{(l)}_{i0} \lambda_k^{(l)} \lambda_{k-1}^{(l)}}{\lambda_k^{(l)} \cdots \lambda_{k-l}^{(l)} \prod_{k=1}^l d\lambda_k^{(l)}} \prod_{i=0}^m \prod_{k=1}^n d\lambda_k^{(l)},
\]

with \( C \) a normalization constant such that \( P^{\text{Dyn}}_n = \tau_n|_\mathcal{L} \), where the locus \( \mathcal{L} \) is defined as \( \{ l = 1, 2, \ldots, m-1, k = 1, 2, \ldots, m, c_k = e^{-ik} \} \).

In the latter part of this paper, the phrase “variables are on the locus \( \mathcal{L} \)” means that \( t^{(l)}_i \) and \( c^{(l)}_{ij} \) are given by (16).

Remark 3. In the latter part of the paper, we often regard \( P^{\text{Dyn}}_n \) as a function with variables \( t^{(l)}_i \) and \( c^{(l)}_{ij} \), and parameters \( a^{(l)}_i \), though most of the variables are 0, according to (16). Therefore it is legitimate to consider \( \frac{\partial}{\partial t^{(l)}_i} P^{\text{Dyn}}_n \) etc.

Remark 4. Since we allow \( s^{(l)}_i \) to be \(-\infty\) and \( a^{(l)}_{2l} \) to be \(+\infty\), the integral in (15) may be divergent for general values of \( t^{(l)}_i \) and \( c^{(l)}_{ij} \). However, if we assume \( t^{(l)}_i = 0 \) for \( i > 2 \), \( c^{(l)}_{ij} = 0 \) for \( \max(i, j) > 1 \), and values of \( t^{(l)}_i, t^{(l)}_j \) and \( c^{(l)}_{ij} \) are near the locus \( \mathcal{L} \), then the integral is convergent, and all algebraic operations in latter part of the paper can be taken in this restricted setting, so they are legitimate.

Now we consider actions of \( D^{l, 1} \) on \( \tau_n \). Since \( D^{l, 1} \) acts on the integral domains of \( \lambda^{(l)}_1, \ldots, \lambda^{(l)}_n \), by the formula

\[
\left[ \frac{\partial}{\partial a} + \frac{\partial}{\partial b} \right] \int_a^b f(x) \, dx = f(b) - f(a) = \int_a^b f'(x) \, dx,
\]

we get

\[
D^{l+1, 1} \tau_n = \frac{1}{C} \sum_{i=1}^{2n} \left[ \frac{\partial}{\partial a^{(l)}_i} + \frac{\partial}{\partial b^{(l)}_i} \right] \int \cdots \int_{U(0)^n \times \cdots \times U(m)^n} \left( V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{i=0}^m \prod_{k=1}^n \sum_{k=1}^\infty \frac{\partial}{\partial \lambda_k^{(l)}} \left( V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{i=0}^m \prod_{k=1}^n \sum_{k=1}^\infty \frac{\partial}{\partial \lambda_k^{(l)}} \left( V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{i=0}^m \prod_{k=1}^n \sum_{k=1}^\infty \frac{\partial}{\partial \lambda_k^{(l)}} \left( V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{i=0}^m \prod_{k=1}^n \sum_{k=1}^\infty \frac{\partial}{\partial \lambda_k^{(l)}} \left( \frac{\partial}{\partial \lambda_k^{(l)}} \right) \right) \right) \right) \prod_{i=0}^m \prod_{k=1}^n d\lambda_k^{(l)} \right),
\]

\[
= \frac{1}{C} \int \cdots \int_{U(0)^n \times \cdots \times U(m)^n} \left[ \sum_{i=1}^n \sum_{k=1}^\infty \lambda_k^{(l)} \lambda_{k-1}^{(l)} + \sum_{i,j=0}^{\infty} \frac{c^{(l)}_{ij}}{i+j+1} \right] \prod_{i=0}^m \prod_{k=1}^n d\lambda_k^{(l)} \right).
\]
\[
\times \left( V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{l=0}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{l=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} \left( \sum_{i=1}^{n} \frac{c_{ij}^{(l)}}{\partial \lambda^{(l)}_{i,j}} \right) \right) \prod_{l=0}^{m} \prod_{k=1}^{n} d\lambda^{(l)}_{k} \\
= \frac{1}{c} \int \cdots \int_{U^{(n)} \times \cdots \times U^{(m)}} \left[ n_{t_{1}}^{(0)} + \sum_{i=2}^{\infty} \frac{c_{1}^{(0)}}{\partial t_{i-1}} + \sum_{i=1}^{\infty} \frac{c_{i-1}^{(1)}}{\partial t_{i}^{(1)}} + \sum_{i=2}^{\infty} \frac{c_{i-1}^{(1)}}{\partial t_{i}^{(1)}} \right] \\
\times \left( V(\lambda^{(0)}) V(\lambda^{(m)}) \prod_{l=0}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{l=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{n} \left( \sum_{i=1}^{n} \frac{c_{ij}^{(l)}}{\partial \lambda^{(l)}_{i,j}} \right) \right) \prod_{l=0}^{m} \prod_{k=1}^{n} d\lambda^{(l)}_{k} \\
= \left[ n_{t_{1}}^{(0)} + \sum_{i=2}^{\infty} \frac{c_{1}^{(0)}}{\partial t_{i-1}} + \sum_{i=1}^{\infty} \frac{c_{i-1}^{(1)}}{\partial t_{i}^{(1)}} + \sum_{i=2}^{\infty} \frac{c_{i-1}^{(1)}}{\partial t_{i}^{(1)}} \right] \tau_{n}. \tag{17}
\]

and similarly \((l = 1, \ldots, m - 1)\)

\[
D^{l-1} \tau_{n} = \left[ n_{t_{1}}^{(0)} + \sum_{i=2}^{\infty} \frac{c_{1}^{(0)}}{\partial t_{i-1}} + \sum_{i=1}^{\infty} \frac{c_{i-1}^{(1)}}{\partial t_{i}^{(1)}} + \sum_{i=2}^{\infty} \frac{c_{i-1}^{(1)}}{\partial t_{i}^{(1)}} \right] \tau_{n}. \tag{18}
\]

As explained in Remark 3, we regard \(\tau_{n}^{Dw} \) as a specialization of \(\tau_{n}\), such that the values of \(t_{1}^{(l)}\) and \(c_{ij}^{(l)}\) are given by the locus \(\mathcal{L}\) as in (16), and we get (We abbreviate \(\tau_{n}^{Dw} \) as \(\tau_{n}\) here and later.)

\[
D^{0,1} \tau_{n} = \left[ \frac{2 c_{t}}{1 - c_{t}^{2}} \frac{\partial}{\partial t_{0}} + \frac{2 c_{1}}{1 - c_{1}^{2}} \frac{\partial}{\partial t_{1}^{(2)}} \right] \tau_{n}, \tag{19}
\]

\[
D^{l} \tau_{n} = \left[ \frac{2 c_{1}}{1 - c_{1}^{2}} \frac{\partial}{\partial t_{1}^{(l-1)}} - \left( \frac{2 c_{1}}{1 - c_{1}^{2}} + \frac{2 c_{2}}{1 - c_{2}^{2}} \right) \frac{\partial}{\partial t_{1}^{(1)}} + \frac{2 c_{2}}{1 - c_{2}^{2}} \frac{\partial}{\partial t_{1}^{(1)}} \right] \tau_{n}, \tag{20}
\]

\[
D^{m,1} \tau_{n} = \left[ \frac{2 c_{m}}{1 - c_{m}^{2}} \frac{\partial}{\partial t_{1}^{(m-1)}} - \frac{2 c_{m}}{1 - c_{m}^{2}} \frac{\partial}{\partial t_{1}^{(m)}} \right] \tau_{n}. \tag{21}
\]

Now we define an \((m + 1) \times (m + 1)\) matrix

\[
J = \begin{pmatrix}
2 t_{0}^{(0)} & c_{1,1}^{(1)} & c_{1,1}^{(2)} & \cdots & c_{1,1}^{(m)} \\
c_{1,1}^{(1)} & 2 t_{1}^{(1)} & c_{1,2}^{(1)} & \cdots & c_{1,2}^{(m)} \\
c_{1,2}^{(2)} & c_{1,2}^{(1)} & \cdots & \cdots & c_{1,2}^{(m)} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c_{1,1}^{(m)} & c_{1,2}^{(m)} & \cdots & \cdots & 2 t_{2}^{(m)}
\end{pmatrix}^{-1},
\]

whose rows and columns are indexed from 0 to \(m\). When \(t_{i}^{(l)}\) and \(c_{ij}^{(l)}\) are on the locus \(\mathcal{L}\) as in (16), \(J\) becomes

\[
J_{\mathcal{L}} = \begin{pmatrix}
- \frac{2}{1 - c_{1}^{2}} & \frac{2 c_{1}}{1 - c_{1}^{2}} & \frac{2 c_{2}}{1 - c_{2}^{2}} & \frac{2 c_{3}}{1 - c_{3}^{2}} & \cdots \\
\frac{2 c_{2}}{1 - c_{2}^{2}} & - \frac{2}{1 - c_{1}^{2}} - \frac{2 c_{2}}{1 - c_{2}^{2}} & \frac{2 c_{3}}{1 - c_{3}^{2}} & \frac{2 c_{4}}{1 - c_{4}^{2}} & \cdots \\
\frac{2 c_{3}}{1 - c_{3}^{2}} & \frac{2 c_{2}}{1 - c_{2}^{2}} & - \frac{2}{1 - c_{1}^{2}} - \frac{2 c_{2}}{1 - c_{2}^{2}} & \frac{2 c_{3}}{1 - c_{3}^{2}} & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{2 c_{m}}{1 - c_{m}^{2}} & \frac{2 c_{m-1}}{1 - c_{m-1}^{2}} & \cdots & \cdots & \frac{2 c_{m}}{1 - c_{m}^{2}}
\end{pmatrix}^{-1}.
\]

We can find the entries of the first and the last row of \(J\) explicitly when the variables are on the locus \(\mathcal{L}\):

\[
J_{0,1} = - \frac{1}{2} \prod_{i=1}^{m} c_{i} = - \frac{1}{2} e^{-u}, \tag{22}
\]

\[
J_{m,1} = - \frac{1}{2} \prod_{i=1}^{m-1} c_{m-i+1} = - \frac{1}{2} e^{u-tn}, \tag{23}
\]
and especially
\[ J_{0,m|z} = J_{m,0|z} = -\frac{1}{2} \sum_{i=1}^{m} c_i = -\frac{1}{2} e^{-t_n}. \]

Then let
\[
\begin{pmatrix}
E^{0,1}_{m,1} \\
E^{1,1}_{m,1} \\
\vdots \\
E^{m-1}_{m,1}
\end{pmatrix} = J
\begin{pmatrix}
D^{0,1}_{m,1} \\
D^{1,1}_{m,1} \\
\vdots \\
D^{m-1}_{m,1}
\end{pmatrix},
\]
and we have

**Lemma 1.** When the variables \( t^{(i)}_i \) and \( c^{(j)}_{ij} \) are on the locus \( \mathcal{L} \) as in (16), we have
\[
E^{0,1}E^{m-1}_{m,1} \log \mathbb{P}_n = E^{m-1}_{m,1}E^{0,1} \log \mathbb{P}_n = \frac{\partial^2 \log \mathbb{P}_n}{\partial t^{(0)}_1 \partial t^{(m)}_1} - \frac{n}{2} e^{-t_n}.
\]

**Proof.** First, since \( E^{0,1} \) and \( E^{m-1}_{m,1} \) are linear combinations of \( D^{i,1} \)'s, they are differential operators of order 1, and we have
\[
E^{0,1}E^{m-1}_{m,1} \log \mathbb{P}_n = -\frac{E^{0,1}_{m,1}E^{m-1}_{m,1} \mathbb{P}_n}{\mathbb{P}_n} + \frac{E^{0,1}_{m,1}E^{m-1}_{m,1} \mathbb{P}_n}{\mathbb{P}_n}.
\]
By (19)–(23), when variables are on the locus \( \mathcal{L} \), we get for \( l = 0, \ldots, m \)
\[
E^{l,1}_{m,1} \mathbb{P}_n = \left[ \sum_{i=0}^{m} J_{l,i} D^{i,1}_{m,1} \right] \mathbb{P}_n = \frac{\partial \mathbb{P}_n}{\partial t^{(l)}_1}.
\]
Therefore when variables are on the locus \( \mathcal{L} \),
\[
E^{0,1}E^{m-1}_{m,1} \log \mathbb{P}_n = -\frac{\partial^2 \log \mathbb{P}_n}{\partial t^{(0)}_1 \partial t^{(m)}_1} + \frac{E^{0,1}_{m,1} \mathbb{P}_n}{\mathbb{P}_n}.
\]

Here we need to be careful about the term \( E^{0,1}_{m,1} \frac{\partial}{\partial t^{(m)}_1} \mathbb{P}_n \). By (17) and (18), the action of \( E^{m-1}_{m,1} \) on \( \mathbb{P}_n \) is equivalent to that of a differential operator of the form \( \frac{\partial}{\partial t^{(m)}_1} + \cdots \), which does not contain \( t^{(m)}_1 \) explicitly. When variables are on the locus \( \mathcal{L} \), all terms of the differential operator vanish except for \( \frac{\partial}{\partial t^{(m)}_1} \), so we can ignore them and replace \( E^{m-1}_{m,1} \mathbb{P}_n \) by \( \frac{\partial}{\partial t^{(m)}_1} \) between \( E^{0,1}_{m,1} \) and \( \mathbb{P}_n \).

Since \( E^{0,1}_{m,1} \) and \( \frac{\partial}{\partial t^{(m)}_1} \) commute,
\[
E^{0,1}_{m,1} \frac{\partial}{\partial t^{(m)}_1} \mathbb{P}_n = \frac{\partial}{\partial t^{(m)}_1} E^{0,1}_{m,1} \mathbb{P}_n.
\]

**By (17), (18), (22) and (24), we have the identity for the action of \( E^{0,1}_{m,1} \) on \( \tau_n \)
\[
E^{0,1}_{m,1} \tau_n = J_{0,0} \left[ n \tau^{(0)}_1 + \sum_{i=2}^{\infty} \frac{\partial}{\partial t^{(0)}_1} \right] \tau_n
+ \sum_{l=1}^{m} J_{0,l} \left[ n \tau^{(l)}_1 + \sum_{i=2}^{\infty} \frac{\partial}{\partial t^{(l)}_1} \right] \tau_n
+ \sum_{l=1}^{m} \sum_{i=1}^{\infty} \frac{\partial}{\partial c^{(l)}_{i-1,j}} \tau_n
+ \sum_{l=1}^{m} \sum_{j=1}^{\infty} \frac{\partial}{\partial c^{(l)}_{i-1,j}} \tau_n
+ J_{0,m} e^{-t_n} \left[ n \tau^{(m)}_1 + \sum_{i=2}^{\infty} \frac{\partial}{\partial t^{(m)}_1} \right] \tau_n
+ \sum_{l=1}^{m} \sum_{j=1}^{\infty} \frac{\partial}{\partial c^{(m)}_{i-1,j}} \tau_n
+ \sum_{l=1}^{m} \sum_{j=1}^{\infty} \frac{\partial}{\partial c^{(m)}_{i-1,j}} \tau_n
= \left[ \frac{\partial}{\partial t^{(m)}_1} + n J_{0,m} \tau^{(m)}_1 \right] \tau_n,
\]
such that all terms except for \( \frac{\partial}{\partial t^{(m)}_1} + n J_{0,m} \tau^{(m)}_1 \) in the operator on the right-hand side of (30) do not contain \( t^{(m)}_1 \) and vanish when variables are on the locus \( \mathcal{L} \). So when variables are on the locus \( \mathcal{L} \),
\[
\frac{\partial}{\partial t^{(m)}_1} E^{0,1}_{m,1} \mathbb{P}_n = \frac{\partial}{\partial t^{(m)}_1} \left[ \frac{\partial}{\partial t^{(m)}_1} + n J_{0,m} \tau^{(m)}_1 \right] \mathbb{P}_n = \frac{\partial^2 \mathbb{P}_n}{\partial t^{(0)}_1 \partial t^{(m)}_1} + n J_{0,m} \tau^{(m)}_1 \mathbb{P}_n.
\]
\[ E_{21}^{m,1} \log \Pi_n = -\frac{\partial^2 \Pi_n}{\partial t_{11}^{\text{out} (m)} \partial t_{11}^{\text{in} (m)}} + \frac{\partial \Pi_n}{\partial t_{11}^{\text{out} (m)}} + n \log \Pi_n = \frac{\partial^2 \log \Pi_n}{\partial t_{11}^{\text{out} (m)} \partial t_{11}^{\text{in} (m)}} - \frac{n}{2} e^{-\tau_n}. \]

Similarly to (17) and (18), with the help of the formula
\[ \left[ \frac{d}{dx} + \frac{b}{dx} \right] \int_a^b f(x) \, dx = b f(b) - a f(a) = \int_a^b (xf(x))' \, dx, \]
we get \( \left[ \sum_{k=1}^n \frac{\partial}{\partial \lambda_k^{(0)}} \right] \) is regarded as an operator
\[
D^{0,2} \tau_n = \frac{1}{C} \left[ \sum_{i=1}^n \left( \frac{\partial^2 a^{(0)}}{\partial t_{11}^{\text{out} (m)} \partial t_{11}^{\text{in} (m)}} \left( V(\lambda_k^{(0)}) \right) \right) \sum_{k=1}^n \left( \frac{\partial}{\partial \lambda_k^{(0)}} \lambda_k^{(0)} \right) \right] \prod_{i=1}^n \prod_{k=1}^n d \lambda_k^{(i)}
\]
\[
= \frac{1}{C} \left[ \sum_{i=1}^n \left( \frac{\partial}{\partial t_{11}^{\text{out} (m)}} + \sum_{j=1}^m \frac{\partial^2 c_{ij}^{(1)}(l)}{\partial t_{11}^{\text{out} (m)} \partial t_{11}^{\text{in} (m)}} \right) \left( V(\lambda_k^{(0)}) \right) \sum_{k=1}^n \left( \frac{\partial}{\partial \lambda_k^{(0)}} \lambda_k^{(0)} \right) \right] \prod_{i=1}^n \prod_{k=1}^n d \lambda_k^{(i)}
\]
\[
= \frac{1}{C} \left[ \sum_{i=1}^n \left( \frac{\partial}{\partial t_{11}^{\text{out} (m)}} + \sum_{j=1}^m \frac{\partial^2 c_{ij}^{(1)}(l)}{\partial t_{11}^{\text{out} (m)} \partial t_{11}^{\text{in} (m)}} \right) \left( V(\lambda_k^{(0)}) \right) \sum_{k=1}^n \left( \frac{\partial}{\partial \lambda_k^{(0)}} \lambda_k^{(0)} \right) \right] \prod_{i=1}^n \prod_{k=1}^n d \lambda_k^{(i)}
\]
\[
= \left[ \sum_{i=1}^n \frac{\partial}{\partial t_{11}^{\text{out} (m)}} + \sum_{j=1}^m \frac{\partial^2 c_{ij}^{(1)}(l)}{\partial t_{11}^{\text{out} (m)} \partial t_{11}^{\text{in} (m)}} \right] \tau_n,
\]
and similarly
\[
D^{m,2} \tau_n = \left[ \sum_{i=1}^n \frac{\partial}{\partial t_{11}^{\text{out} (m)}} + \sum_{j=1}^m \frac{\partial^2 c_{ij}^{(m)}(l)}{\partial t_{11}^{\text{out} (m)} \partial t_{11}^{\text{in} (m)}} \right] \tau_n,
\]
and for \( l = 1, \ldots, m - 1 \)
\[
D^{l,2} \tau_n = \left[ \sum_{i=1}^n \frac{\partial}{\partial t_{11}^{\text{out} (m)}} + \sum_{j=1}^m \frac{\partial^2 c_{ij}^{(l)}(l)}{\partial t_{11}^{\text{out} (m)} \partial t_{11}^{\text{in} (m)}} \right] \tau_n.
\]
When variables are on the locus \( \mathcal{L} \) we get (\( l = 1, \ldots, m - 1 \))
\[
D^{0,2} \varphi_n = \left[ -\frac{2 \lambda}{1 - \lambda} \right] \varphi_n,
\]
\[
D^{l,2} \varphi_n = \left[ -\frac{2 \lambda^2}{1 - \lambda} \right] \varphi_n,
\]
\[
D^{m,2} \varphi_n = \left[ -\frac{2 \lambda^3}{1 - \lambda} \right] \varphi_n.
\]
If we define (\( l = 1, 2, \ldots, m - 1 \))
\[
E_{0,2} = D^{0,2} - c_{11}^{(1)} \frac{\partial}{\partial c_{11}^{(1)}},
\]
\[
E_{l,2} = D^{l,2} - c_{11}^{(l)} \frac{\partial}{\partial c_{11}^{(l)}},
\]
\[
E_{m,2} = D^{m,2} - c_{11}^{(m)} \frac{\partial}{\partial c_{11}^{(m)}},
\]
we have
Lemma 2. For \( k, l = 0, 1, \ldots, m \), when variables are on the locus \( \mathcal{L} \) in (16),

\[
E^{k-2} E^{l} \log \mathbb{P}_n = 2 t^{(k)}_2 \left| \frac{\partial^2 \mathbb{P}_n}{\partial t^{(k)}_2 \partial t^{(l)}_1} \right| + \delta_k E^{l} \log \mathbb{P}_n.
\] (36)

Proof. With arguments similar to those for (26) and (28), we get

\[
E^{k-2} E^{l} \log \mathbb{P}_n = \frac{E^{k-2} \mathbb{P}_n E^{l} \mathbb{P}_n}{\mathbb{P}_n} + \frac{E^{k-2} E^{l} \mathbb{P}_n}{\mathbb{P}_n} = - \left( 2 t^{(k)}_2 \left| \frac{\partial \mathbb{P}_n}{\partial t^{(k)}_2} \right| + C \mathbb{P}_n \right) \frac{\partial \mathbb{P}_n}{\partial t^{(l)}_1} + \frac{E^{k-2} \frac{\partial}{\partial t^{(l)}_1} \mathbb{P}_n}{\mathbb{P}_n},
\]

with

\[
C = \begin{cases} \frac{n(n+1)}{2} & k = 1 \text{ or } m, \\ n & \text{otherwise.} \end{cases}
\]

Similar to (29) and (30), we have

\[
E^{k-2} \frac{\partial}{\partial t^{(l)}_1} \mathbb{P}_n = \frac{\partial}{\partial t^{(l)}_1} E^{k-2} \mathbb{P}_n
\]
and

\[
E^{k-2} \tau_n = \left( 2 t^{(k)}_2 \frac{\partial}{\partial t^{(k)}_2} + t^{(l)}_1 \frac{\partial}{\partial t^{(l)}_1} + C + \cdots \right) \tau_n,
\] (37)

such that all terms except for \( 2 t^{(k)}_2 \frac{\partial}{\partial t^{(k)}_2} + t^{(l)}_1 \frac{\partial}{\partial t^{(l)}_1} + C \) in the operator on the right-hand side of (37) do not contain \( t^{(l)}_1 \) explicitly and vanish when variables are on the locus \( \mathcal{L} \). Therefore with an argument similar to that for (31), when variables are on \( \mathcal{L} \) we have

\[
E^{k-2} E^{l} \log \mathbb{P}_n = - \left( 2 t^{(k)}_2 \left| \frac{\partial \mathbb{P}_n}{\partial t^{(k)}_2} \right| + C \mathbb{P}_n \right) \frac{\partial \mathbb{P}_n}{\partial t^{(l)}_1} + \frac{E^{k-2} \frac{\partial}{\partial t^{(l)}_1} \mathbb{P}_n}{\mathbb{P}_n} = 2 t^{(k)}_2 \left| \frac{\partial^2 \mathbb{P}_n}{\partial t^{(k)}_2 \partial t^{(l)}_1} \right| + \frac{E^{k-2} \frac{\partial}{\partial t^{(l)}_1} \mathbb{P}_n}{\mathbb{P}_n},
\]

since when variables are on \( \mathcal{L} \)

\[
E^{l} \log \mathbb{P}_n = \frac{E^{l} \mathbb{P}_n}{\mathbb{P}_n} = \frac{\partial \mathbb{P}_n}{\partial t^{(l)}_1} \mathbb{P}_n = \frac{\partial \mathbb{P}_n}{\partial s_l} \mathbb{P}_n. \quad \square
\]

When variables are on the locus \( \mathcal{L} \), as defined in (16), \( c^{(k)}_{1,1} \) and \( t^{(l)}_1 \) are functions of \( c_1 = e^{-s_1}, \ldots, c_n = e^{-s_m} \) and all other variables are 0. Now we regard \( \mathbb{P}_n \) as a function in variables \( s_1, \ldots, s_m \), and by the chain rule we get as operators on \( \mathbb{P}_n \) \( (l = 1, 2, \ldots, m) \)

\[
\frac{\partial}{\partial s_i} = \frac{2 c_i^2}{1 - c_i^2} \frac{\partial}{\partial t^{(l)}_2} + \frac{2 c_i^2}{1 - c_i^2} \frac{\partial}{\partial t^{(l)}_1},
\]
and

\[
c^{(l)}_{1,1} \frac{\partial}{\partial t^{(l)}_1} = \frac{2 c_i^2}{1 - c_i^2} \frac{\partial}{\partial t^{(l)}_2} + \frac{2 c_i^2}{1 - c_i^2} \frac{\partial}{\partial t^{(l)}_1} - \frac{1 - c_i^2}{1 + c_i^2} \frac{\partial}{\partial s_l}.
\]

Therefore by (33)–(35) we get that as operators acting on \( \mathbb{P}_n \) \( (l = 1, 2, \ldots, m - 1) \)

\[
E^{0,2} = D^{0,2} - \frac{2 c_i^2}{1 - c_i^2} \frac{\partial}{\partial t^{(l)}_2} - \frac{2 c_i^2}{1 - c_i^2} \frac{\partial}{\partial t^{(l)}_1} + \frac{1 - c_i^2}{1 + c_i^2} \frac{\partial}{\partial s_1},
\] (38)

\[
E^{1,2} = D^{1,2} - \frac{2 c_i^2}{1 - c_i^2} \frac{\partial}{\partial t^{(l)}_2} - \left( \frac{2 c_i^2}{1 - c_i^2} + \frac{2 c_i^2}{1 - c_i^2} \frac{\partial}{\partial t^{(l)}_2} \right) + \frac{1 - c_i^2}{1 + c_i^2} \frac{\partial}{\partial s_l} + \frac{1 - c_i^2}{1 + c_i^2} \frac{\partial}{\partial t^{(l)}_2}.
\] (39)

\[
E^{m,2} = D^{m,2} - \frac{2 c_m^2}{1 - c_m^2} \frac{\partial}{\partial t^{(m-1)}_2} - \frac{2 c_m^2}{1 - c_m^2} \frac{\partial}{\partial t^{(m-1)}_2} + \frac{1 - c_m^2}{1 + c_m^2} \frac{\partial}{\partial s_m}.
\] (40)
Now we denote \((l = 1, 2, \ldots, m - 1)\)

\[
F^{0,2} = D^{0,2} + \frac{1 - c_l^2}{1 + c_l^2} \frac{\partial}{\partial s_l},
\]

\[
F^{l,2} = D^{l,2} + \frac{1 - c_l^2}{1 + c_l^2} \frac{\partial}{\partial s_l} + \frac{1 - c_{l+1}^2}{1 + c_{l+1}^2} \frac{\partial}{\partial s_{l+1}},
\]

\[
F^{m,2} = D^{m,2} + \frac{1 - c_m^2}{1 + c_m^2} \frac{\partial}{\partial s_m},
\]

and have

**Lemma 3.** For \(l = 1, \ldots, m - 1,\)

\[
F^{0,2} \log \mathbb{P}_n = \left[ -\frac{2}{1 - c_l^2} \frac{\partial^2}{\partial t_2^{(l)}} + \frac{2c_l^2}{1 - c_l^2} \frac{\partial^2}{\partial t_2^{(l)}} \frac{\partial^2}{\partial t_1^{(l)}} \right] \log \mathbb{P}_n, \tag{41}
\]

\[
F^{l,2} \log \mathbb{P}_n = \left[ \frac{2c_l^2}{1 - c_l^2} \frac{\partial^2}{\partial t_2^{(l)}} - \left( \frac{2}{1 - c_l^2} + \frac{2c_{l+1}^2}{1 - c_{l+1}^2} \right) \frac{\partial^2}{\partial t_2^{(l)}} \frac{\partial^2}{\partial t_1^{(l)}} + \frac{2c_{l+1}^2}{1 - c_{l+1}^2} \frac{\partial^2}{\partial t_2^{(l)}} \frac{\partial^2}{\partial t_1^{(l)}} \right] \log \mathbb{P}_n, \tag{42}
\]

\[
F^{m,2} \log \mathbb{P}_n = \left[ \frac{2c_m^2}{1 - c_m^2} \frac{\partial^2}{\partial t_2^{(m)}} - \frac{2}{1 - c_m^2} \frac{\partial^2}{\partial t_2^{(m)} \partial t_1^{(m)}} \right] \log \mathbb{P}_n + E^{m,1} \log \mathbb{P}_n. \tag{43}
\]

**Proof.** We only prove (41) and (43), and (42) can be proved similarly. By (38) and (40), as operators acting on \(\mathbb{P}_n\), we have

\[
F^{0,2} = F^{0,2} + \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} + \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} \frac{\partial}{\partial s_l}, \tag{44}
\]

\[
F^{m,2} = F^{m,2} + \frac{2c_m^2}{1 - c_m^2} \frac{\partial}{\partial t_2^{(m)}} + \frac{2c_m^2}{1 - c_m^2} \frac{\partial}{\partial t_2^{(m)}} \frac{\partial}{\partial s_m}. \tag{45}
\]

Similar to (28) and (31), we have

\[
\frac{\partial}{\partial t_2^{(l)}} E^{1,1} \mathbb{P}_n = - \frac{\partial}{\partial t_2^{(l)}} \frac{\partial}{\partial t_2^{(l)}} E^{1,1} \mathbb{P}_n + \frac{\partial}{\partial t_2^{(l)}} \frac{\partial^2}{\partial t_2^{(l)} \partial t_1^{(l)}} \log \mathbb{P}_n,
\]

Thus with the results of (27), (36), (44) and (45) we have

\[
F^{0,2} \log \mathbb{P}_n = E^{0,2} \log \mathbb{P}_n + \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} E^{1,1} \log \mathbb{P}_n + \frac{2c_l^2}{1 - c_l^2} \frac{\partial}{\partial t_2^{(l)}} E^{m,1} \log \mathbb{P}_n
\]

\[
= 2t_2^{(l)} \left[ \frac{\partial^2}{\partial t_2^{(l)} \partial t_1^{(l)}} \log \mathbb{P}_n + \frac{2c_l^2}{1 - c_l^2} \frac{\partial^2}{\partial t_2^{(l)} \partial t_1^{(l)}} \right] \log \mathbb{P}_n,
\]

and

\[
F^{m,2} \log \mathbb{P}_n = E^{m,2} \log \mathbb{P}_n + \frac{2c_m^2}{1 - c_m^2} \frac{\partial}{\partial t_2^{(m)}} E^{1,1} \log \mathbb{P}_n + \frac{2c_m^2}{1 - c_m^2} \frac{\partial}{\partial t_2^{(m)}} E^{m,1} \log \mathbb{P}_n
\]

\[
= \left( 2t_2^{(m)} \frac{\partial^2}{\partial t_2^{(m)} \partial t_1^{(m)}} + E^{m,1} \log \mathbb{P}_n \right) + \frac{2c_m^2}{1 - c_m^2} \frac{\partial^2}{\partial t_2^{(m)} \partial t_1^{(m)}} \log \mathbb{P}_n
\]

\[
= \left[ -\frac{2}{1 - c_m^2} \frac{\partial^2}{\partial t_2^{(m)} \partial t_1^{(m)}} + \frac{2c_m^2}{1 - c_m^2} \frac{\partial^2}{\partial t_2^{(m)} \partial t_1^{(m)}} \right] \log \mathbb{P}_n + E^{m,1} \log \mathbb{P}_n. \]

Finally we define

\[
\begin{pmatrix}
C^{0,2} \\
C^{l,2} \\
\vdots \\
C^{m,2}
\end{pmatrix} = K \begin{pmatrix}
F^{0,2} \\
F^{l,2} \\
\vdots \\
F^{m,2}
\end{pmatrix},
\]
where
\[
K = \begin{pmatrix} -\frac{2}{1-c_1^2} & -\frac{2c_1^2}{1-c_1^4} & \ldots & -\frac{2c_1^{2m-2}}{1-c_1^{2m}} \\
\frac{1}{1-c_1^2} & \frac{2}{1-c_1^4} & \ldots & \frac{2c_1^{2m-1}}{1-c_1^{2m}} \\
\frac{1}{1-c_1^2} & \frac{1}{1-c_1^4} & \ldots & \frac{1}{1-c_1^{2m}} \\
\ldots & \ldots & \ldots & \ldots \\
\frac{1}{1-c_1^2} & \frac{1}{1-c_1^4} & \ldots & \frac{1}{1-c_1^{2m}} \\
\frac{2c_1^{2m}}{1-c_1^{2m}} & \frac{2c_1^{2m}}{1-c_1^{2m}} & \ldots & \frac{2c_1^{2m}}{1-c_1^{2m}} \\
\end{pmatrix}^{-1}.
\]

We can get \(K^{-1}\) by substituting each \(c_i\) in \(J^{-1}\) by \(c_i^2\), so we have
\[
K_{0,l} = -\frac{1}{2} \prod_{i=1}^{l} c_i^2, \quad K_{m,l} = -\frac{1}{2} \prod_{i=1}^{m-l} c_{m-i+1}^2
\]
and get by (41)-(43).

**Lemma 4.**
\[
G^{0.2}E^{m.1}\log P_n = \frac{\partial^2}{\partial t_2^{(0)} \partial t_1^{(m)}} \log P_n + K_{0,m}E^{m.1}\log P_n.
\] (46)

Symmetrically, we can get by the same method

**Lemma 5.**
\[
G^{m.2}E^{0.1}\log P_n = \frac{\partial^2}{\partial t_2^{(m)} \partial t_1^{(0)}} \log P_n + K_{m,0}E^{0.1}\log P_n.
\] (47)

By the result of [6]
\[
\frac{\partial}{\partial t_0^{(0)}} \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\mu^2}{\mu_{11}^{(0)} \mu_{22}^{(0)}} \log \tau_n,
\]
\[
\frac{\partial}{\partial t_0^{(m)}} \log \frac{\tau_{n+1}}{\tau_{n-1}} = \frac{\mu^2}{\mu_{11}^{(m)} \mu_{22}^{(m)}} \log \tau_n,
\]

and by (27), which implies the differential equation when variables of \(\tau_{n-1}\) and \(\tau_{n+1}\) are on the locus \(\mathcal{L}\)
\[
\left. \frac{\partial}{\partial t_0^{(0)}} \log \frac{\tau_{n+1}}{\tau_{n-1}} \right|_{\mathcal{L}} = E^{0.1} \log \frac{\tau_{n+1}}{\tau_{n-1}} |_{\mathcal{L}},
\] (48)

we get the differential equation with respect to parameters \(a^{(0)}_n\), when variables of \(\tau_{n-1}\), \(\tau_n\), and \(\tau_{n+1}\) are on the locus \(\mathcal{L}\), by (25) and (46)-(48)
\[
E^{0.1} \log \frac{\tau_{n+1}}{\tau_{n-1}} |_{\mathcal{L}} = G^{0.2}E^{m.1} \log \tau_n |_{\mathcal{L}} - K_{0,m}E^{m.1} \log \tau_n |_{\mathcal{L}},
\]
\[
E^{m.1} \log \frac{\tau_{n+1}}{\tau_{n-1}} |_{\mathcal{L}} = G^{m.2}E^{0.1} \log \tau_n |_{\mathcal{L}} - K_{m,0}E^{0.1} \log \tau_n |_{\mathcal{L}}.
\]

By the identity
\[
E^{0.1}E^{m.1} \log \frac{\tau_{n+1}}{\tau_{n-1}} |_{\mathcal{L}} = E^{m.1}E^{0.1} \log \frac{\tau_{n+1}}{\tau_{n-1}} |_{\mathcal{L}},
\]

we get the final result
\[
E^{0.1}G^{m.2}E^{0.1} \log \tau_n |_{\mathcal{L}} - K_{m,0}E^{0.1} \log \tau_n |_{\mathcal{L}} = E^{m.1}G^{0.2}E^{m.1} \log \tau_n |_{\mathcal{L}} - K_{0,m}E^{m.1} \log \tau_n |_{\mathcal{L}},
\] (49)
Now we denote specializations of differential operators when variables are on the locus \( \Sigma \)
\[
\mathcal{A}_1 = -2E^{0,1}, \\
\mathcal{B}_1 = -2E^{m,1}, \\
\mathcal{A}_2 = -2(G^{0,2} - K_{0,m}), \\
\mathcal{B}_2 = -2(G^{m,2} - K_{m,0}),
\]
which agree with (7)–(10), and we get the Eq. (11), after the notation change \( p_n^{\text{Dyson}} \to t_m|_\Sigma \).

**Remark 5.** In the 2-time case, i.e. \( m = 1 \),
\[
\mathcal{A}_1 = D^{0,1} + c_1 D^{1,1}, \\
\mathcal{B}_1 = c_1 D^{0,1} + D^{1,1}, \\
\mathcal{A}_2 = G^{0,2} + c_2 F^{1,2} - c_1^2 = D^{0,1} + c_1^2 D^{1,1} + (1 - c_1^2) \frac{\partial}{\partial t_1} - c_1^2, \\
\mathcal{B}_2 = c_1^2 G^{0,2} + F^{1,2} - c_1^2 = c_1^2 D^{0,1} + D^{1,1} + (1 - c_1^2) \frac{\partial}{\partial t_1} - c_1^2.
\]

Our PDE (11) agrees with that in [5].

3. The joint probability in the Airy process

In this section we adapt notations defined in (3)–(5), and by **Remark 1**, \( a^{(i)}_m = \bar{a}^{(i)}_m = -\infty \). We denote \( l = 0, 1, \ldots, m \)
\[
\bar{D}^{l,1} = \sum_{k=1}^{2\eta} \frac{\partial}{\partial a^{(i)}_k}, \quad \bar{D}^{l,2} = \sum_{k=1}^{2\eta} a^{(i)}_k \frac{\partial}{\partial a^{(i)}_k},
\]
if all \( a^{(i)}_m \to +\infty \), otherwise drop the \( a^{(i)}_m \) part. We can write the differential operators defined for the Dyson process as
\[
\mathcal{A}_1 = \sqrt{2\eta} \sum_{l=0}^{m} e^{-i\bar{a}^{(i)}_l \bar{D}^{l,1}}, \\
\mathcal{B}_1 = \sqrt{2\eta} \sum_{l=0}^{m} e^{\bar{a}^{(i)}_l \bar{D}^{l,1}},
\]
\[
\mathcal{A}_2 = \sum_{l=0}^{m} e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,2}} + 2\eta^2 \sum_{l=0}^{m} e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,1} + \bar{n}} \sum_{l=1}^{m} (1 - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,1}} - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,2}} \frac{\partial}{\partial t_1} - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,1}}),
\]
\[
\mathcal{B}_2 = \sum_{l=0}^{m} e^{2i\bar{a}^{(i)}_l \bar{D}^{l,2}} + 2\eta^2 \sum_{l=0}^{m} e^{2i\bar{a}^{(i)}_l \bar{D}^{l,1} + \bar{n}} \sum_{l=1}^{m} (e^{2i\bar{a}^{(i)}_l \bar{D}^{l,1}} - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,2}} \frac{\partial}{\partial t_1} - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,1}}).
\]

It is not difficult to see that (11) implies
\[
[A_1 B_2 A_1 - B_1 A_2 B_1] \log \mathcal{P}_n \cdot (A_1 B_1 A_1 \log \mathcal{P}_n + 2n e^{-t_m}) = B_2 A_1 \log \mathcal{P}_n \cdot A_1 B_1 A_1 \log \mathcal{P}_n - A_2 B_1 \log \mathcal{P}_n \cdot B_1 A_1 B_1 \log \mathcal{P}_n.
\]
Substituting (50)–(53) into (54), we get
\[
\left( \left[ \sum_{l=0}^{m} e^{-i\bar{D}^{l,1}} \right] \left[ \sum_{l=0}^{m} e^{2i\bar{a}^{(i)}_l \bar{D}^{l,2}} + 2\bar{n}^2 \sum_{l=0}^{m} e^{2i\bar{a}^{(i)}_l \bar{D}^{l,1}} \left[ \sum_{l=0}^{m} e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,1}} \right] \log \mathcal{P}_n \right) \right.
\]
\[
- \left[ \sum_{l=0}^{m} e^{i\bar{a}^{(i)}_l \bar{D}^{l,1}} \right] \left[ \sum_{l=0}^{m} e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,2}} + 2\bar{n}^2 \sum_{l=0}^{m} e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,1}} + \bar{n} \sum_{l=1}^{m} (1 - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,1}} - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,2}} \frac{\partial}{\partial t_1} - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,1}}) \left[ \sum_{l=0}^{m} e^{2i\bar{a}^{(i)}_l \bar{D}^{l,1}} \right] \log \mathcal{P}_n \right) \right.
\]
\[
\times \left( \left[ \sum_{l=0}^{m} e^{-i\bar{D}^{l,1}} \right] \left[ \sum_{l=0}^{m} e^{i\bar{a}^{(i)}_l \bar{D}^{l,1}} \right] \log \mathcal{P}_n + \bar{n}^2 e^{-\bar{a}^{(i)}_m} \right) \right) \left[ \sum_{l=0}^{m} e^{2i\bar{a}^{(i)}_l \bar{D}^{l,2}} + 2\bar{n}^2 \sum_{l=0}^{m} e^{2i\bar{a}^{(i)}_l \bar{D}^{l,1}} + \bar{n} \sum_{l=1}^{m} (e^{2i\bar{a}^{(i)}_l \bar{D}^{l,1}} - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,2}} \frac{\partial}{\partial t_1} - e^{-2i\bar{a}^{(i)}_l \bar{D}^{l,1}}) \right.
\]
\[
\times \left[ \sum_{l=0}^{m} e^{-i\bar{D}^{l,1}} \right] \log \mathcal{P}_n \times \left[ \sum_{l=0}^{m} e^{-i\bar{a}^{(i)}_l \bar{D}^{l,1}} \right] \left[ \sum_{l=0}^{m} e^{i\bar{a}^{(i)}_l \bar{D}^{l,1}} \right] \left[ \sum_{l=0}^{m} e^{-i\bar{a}^{(i)}_l \bar{D}^{l,1}} \right] \log \mathcal{P}_n
\]
\[-\left[ \sum_{l=0}^{m} e^{-2\hat{t}_l/\hat{n}} \hat{D}_l^2 + 2\hat{n}^2 \sum_{l=0}^{m} e^{-2\hat{t}_l/\hat{n}} \hat{D}_l \right] + \hat{n} \sum_{l=1}^{m} (1 - e^{-2\hat{t}_l/\hat{n}}) \frac{\partial}{\partial \hat{t}_l} - e^{-2\hat{t}_0/\hat{n}} \right] \times \left[ \sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l \right] \log \mathbb{P}_n \times \left[ \sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l \right] \left[ \sum_{l=0}^{m} e^{\hat{t}_l/\hat{n}} \hat{D}_l \right] \log \mathbb{P}_n. \] (55)

Since we have commutator formulas
\[
\begin{align*}
\hat{n} \left( e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} - e^{-2\hat{t}_0/\hat{n}} \right) \frac{\partial}{\partial \hat{t}_l} &= \sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} - e^{(\hat{t}_l + 2\hat{t}_0)/\hat{n}} \hat{D}_l, \\
\hat{n} \sum_{l=1}^{m} (1 - e^{-2\hat{t}_l/\hat{n}}) \frac{\partial}{\partial \hat{t}_l} &= \sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} - e^{(\hat{t}_l + \hat{t}_0)/\hat{n}} \hat{D}_l, \\
\sum_{l=0}^{m} e^{\hat{t}_l/\hat{n}} \hat{D}_l &= \sum_{l=0}^{m} e^{\hat{t}_l/\hat{n}} \hat{D}_l, \\
\sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l \sum_{l=0}^{m} e^{-2\hat{t}_l/\hat{n}} \hat{D}_l^2 &= \sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l \sum_{l=0}^{m} e^{-2\hat{t}_l/\hat{n}} \hat{D}_l^2, \\
\sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l \sum_{l=0}^{m} e^{-\hat{t}_l/\hat{n}} \hat{D}_l^2 &= \sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l \sum_{l=0}^{m} e^{-\hat{t}_l/\hat{n}} \hat{D}_l^2,
\end{align*}
\]

we can change some orders of operator multiplications in (55) and make some cancellations, so that it becomes
\[
\begin{align*}
\left[ \sum_{l=0}^{m} e^{-\hat{t}_l/\hat{n}} \hat{D}_l \right] \left[ \sum_{l=0}^{m} e^{2(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l^2 + 2\hat{n}^2 \sum_{l=0}^{m} e^{2(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l \right] \log \mathbb{P}_n \\
\times \left[ \sum_{l=0}^{m} e^{\hat{t}_l/\hat{n}} \hat{D}_l \right] \left[ \sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l \right] \log \mathbb{P}_n \\
= \left( \sum_{l=0}^{m} e^{\hat{t}_l/\hat{n}} \hat{D}_l \right) \left[ \sum_{l=0}^{m} e^{2(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l^2 + 2\hat{n}^2 \sum_{l=0}^{m} e^{2(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l \right] \log \mathbb{P}_n \times \left[ \sum_{l=0}^{m} e^{-\hat{t}_l/\hat{n}} \hat{D}_l \right] \log \mathbb{P}_n \\
= \left( \sum_{l=0}^{m} e^{\hat{t}_l/\hat{n}} \hat{D}_l \right) \log \mathbb{P}_n \times \left( \sum_{l=0}^{m} e^{-\hat{t}_l/\hat{n}} \hat{D}_l \right) \log \mathbb{P}_n.
\end{align*}
\]

Since all terms of the PDE involve \( \hat{n} \), we can expand the PDE with respect to \( \hat{n} \), with formulas (* can be 1 or 2)
\[
\begin{align*}
\sum_{l=0}^{m} e^{-\hat{t}_l/\hat{n}} \hat{D}_l &= \sum_{l=0}^{m} \hat{D}_l - \frac{\hat{n}}{\hat{n}} \sum_{l=0}^{m} \hat{D}_l \hat{D}_l + \frac{1}{2\hat{n}^2} \sum_{l=0}^{m} \hat{D}_l^2 \hat{D}_l - \frac{1}{6\hat{n}^3} \sum_{l=0}^{m} \hat{D}_l^2 \hat{D}_l + O \left( \frac{1}{\hat{n}^4} \right), \\
\sum_{l=0}^{m} e^{-2\hat{t}_l/\hat{n}} \hat{D}_l^2 &= \sum_{l=0}^{m} \hat{D}_l^2 - \frac{2\hat{n}}{\hat{n}} \sum_{l=0}^{m} \hat{D}_l^2 \hat{D}_l^2 + \frac{2}{3\hat{n}^2} \sum_{l=0}^{m} \hat{D}_l^2 \hat{D}_l^2 + O \left( \frac{1}{\hat{n}^3} \right), \\
\sum_{l=0}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l &= \sum_{l=0}^{m} \hat{D}_l + \frac{\hat{n}}{\hat{n}} \sum_{l=0}^{m} (\hat{t}_l - \hat{t}_0) \hat{D}_l + \frac{1}{2\hat{n}^2} \sum_{l=0}^{m} (\hat{t}_l - \hat{t}_0)^2 \hat{D}_l^2 + \frac{1}{6\hat{n}^3} \sum_{l=0}^{m} (\hat{t}_l - \hat{t}_0)^2 \hat{D}_l^2 + O \left( \frac{1}{\hat{n}^4} \right), \\
\sum_{l=0}^{m} e^{2(\hat{t}_l - \hat{t}_0)/\hat{n}} \hat{D}_l^2 &= \sum_{l=0}^{m} \hat{D}_l^2 + \frac{2\hat{n}}{\hat{n}} \sum_{l=0}^{m} (\hat{t}_l - \hat{t}_0) \hat{D}_l^2 + \frac{2}{3\hat{n}^2} \sum_{l=0}^{m} (\hat{t}_l - \hat{t}_0)^2 \hat{D}_l^2 + \frac{4}{3\hat{n}^3} \sum_{l=0}^{m} (\hat{t}_l - \hat{t}_0)^3 \hat{D}_l^2 + O \left( \frac{1}{\hat{n}^4} \right), \\
\hat{n} \sum_{l=1}^{m} e^{(\hat{t}_l - \hat{t}_0)/\hat{n}} - e^{2\hat{t}_0/\hat{n}} \frac{\partial}{\partial \hat{t}_l} &= \sum_{l=1}^{m} \hat{t}_l \frac{\partial}{\partial \hat{t}_l} + \frac{2}{\hat{n}} \sum_{l=1}^{m} \hat{t}_l (\hat{t}_l - 2\hat{t}_0) \frac{\partial}{\partial \hat{t}_l} + O \left( \frac{1}{\hat{n}^2} \right), \\
\hat{n} \sum_{l=1}^{m} (1 - e^{-2\hat{t}_l/\hat{n}}) \frac{\partial}{\partial \hat{t}_l} &= \sum_{l=1}^{m} \hat{t}_l \frac{\partial}{\partial \hat{t}_l} - \frac{2}{\hat{n}} \sum_{l=1}^{m} \hat{t}_l^2 \frac{\partial}{\partial \hat{t}_l} + O \left( \frac{1}{\hat{n}^2} \right).
\end{align*}
\]
Although the left-hand side of (55) contains $O(n^4)$ terms, after careful calculation we find all $O(n^4)$, $O(n^3)$ and $O(n^2)$ terms disappear, and the equation becomes

\[
\sum_{l=0}^{m} \bar{D}^{-1} \left[ \sum_{l=0}^{m} (\bar{t}_m - \bar{t}_l) \bar{D}^{1/2} + \sum_{l=0}^{m} (\bar{t}_m - \bar{t}_l)^3 - \bar{t}_l^2 \bar{D}^{1/2} + 2 \sum_{l=1}^{m} \bar{t}_l (\bar{t}_m - \bar{t}_l) \frac{\partial}{\partial \bar{t}_l} \right] \log \mathbb{P}_n
\]

\[
+ \sum_{l=0}^{m} \bar{D}^{-1} \left[ \sum_{l=0}^{m} (2\bar{t}_l - \bar{t}_m) \bar{D}^{1/2} + m \bar{t}_l^2 + (\bar{t}_m - \bar{t}_l)^3 \bar{D}^{1/2} + 2 \sum_{l=1}^{m} \bar{t}_l \frac{\partial}{\partial \bar{t}_l} \right] \log \mathbb{P}_n
\]

\[
+ 2 \sum_{l=0}^{m} \bar{t}_l \bar{D}^{1/2} \left[ \sum_{l=0}^{m} (\bar{t}_m - \bar{t}_l)^2 \bar{D}^{1/2} + \sum_{l=0}^{m} (2\bar{t}_l - \bar{t}_m) \bar{D}^{1/2} \right] \log \mathbb{P}_n
\]

\[
= \left\{ \sum_{l=0}^{m} (2\bar{t}_l - \bar{t}_m) \bar{D}^{1/2} \left[ \sum_{l=0}^{m} \bar{D}^{-1} \right] \log \mathbb{P}_n, \sum_{l=0}^{m} \bar{D}^{-1} \left[ \sum_{l=0}^{m} \bar{D}^{1/2} \right] \log \mathbb{P}_n \right\} \sum_{l=0}^{m} \bar{D}^{1/2} + O \left( \frac{1}{n} \right).
\] (56)

The term $O \left( \frac{1}{n} \right)$ in (56) is a quadratic function in terms of $\log \mathbb{P}_n$ and its derivatives with coefficients $O \left( \frac{1}{n} \right)$. By the definition of $\mathbb{P}_{Airy}$ in (6) and the convergence result in [5], we take the limit $n \to \infty$, and get the PDE (12) after the changing of notations, i.e. cleaning all “bars” for variables and operators.

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References