
Read pages 40-43 in the textbook and then do the following problems.

Problem 1. Consider the plane $\mathbb{R}^2$ and two nonzero vectors $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$. Derive the following formula

$$x_1x_2 + y_1y_2 = \|v_1\| \|v_2\| \cos \theta$$

where $\|v_1\|$ and $\|v_2\|$ are the lengths of the vectors and $\theta \in [0, \pi]$ is the angle between them. In your derivation make reference to the Pythagorean Theorem from Euclidean geometry and trigonometry. The common expression is called the dot or scalar or inner product of the two vectors and is commonly denoted by $v_1 \cdot v_2$. If $v_1 \cdot v_2 = 0$ we say that $v_1$ is orthogonal to $v_2$ or that $v_1$ and $v_2$ are orthogonal to each other. A vector that has length one is called a unit vector.

Answer Express the vectors $v_k, k = 1, 2$ using polar coordinates by

$$v_k = \|v_k\| (\cos \theta_k, \sin \theta_k), \quad k = 1, 2.$$ 

Here the Pythagorean formula and definition of trigonometric functions implies that the length of the vectors is given by $\|v_k\|^2 = x_k^2 + y_k^2, k = 1, 2$ and $(\cos \theta_k)^2 + (\sin \theta_k)^2 = 1$. Then use trigonometric formulas for difference of angles to obtain

$$x_1x_2 + y_1y_2 = r_1r_2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) = r_1r_2 \cos \theta$$

where $\theta := \theta_2 - \theta_1$.

Problem 2. A vector $u$ is said to be parallel to a line $L$ or to a line segment $L$ if there exists $p \in L$ such that $p + u \in L$. A vector $v$ is said to be orthogonal to a line $L$ or to a line segment $L$ if $v \cdot u = 0$ whenever $u$ is parallel to $L$. Let $u, v, w$ be unit vectors that are orthogonal to and point inward to the three sides of an equilateral triangle. Prove that if $z$ is a vector that satisfies $z \cdot u \leq 0, z \cdot v \leq 0, z \cdot w \leq 0$ then $z = 0$. Suggestion: choose a specific equilateral triangle with one side along the horizontal axis and the...
vertex that is opposite this side located at \((0, 1)\). Then compute the coordinates of \(u, v, w\). Then show that each of the four vectors having coordinates \((1, 0), (-1, 0), (0, 1), (0, -1)\) can be expressed as a linear combination having nonnegative coefficients of the vectors \(u, v, w\). Express \(z = (z_1, z_2)\) and then show that \(z_1 \leq 0\) and \(z_1 \geq 0\) and \(z_2 \leq 0\) and \(z_2 \geq 0\). What does this imply about the vector \(z\)?

**Answer** Choose the vertices of the triangle to be \((-\frac{1}{\sqrt{3}}, 0), (\frac{1}{\sqrt{3}}, 0), (0, 1)\), hence

\[
u = (0, 1), \quad v = -\frac{1}{2}(\sqrt{3}, 1), \quad w = \frac{1}{2}(\sqrt{3}, -1).
\]

Therefore

\[
(1, 0) = \frac{1}{\sqrt{3}}u + \frac{2}{\sqrt{3}}w,
\]

\[
(-1, 0) = \frac{2}{\sqrt{3}}u + \sqrt{3}v + \frac{1}{\sqrt{3}}w,
\]

\[
(0, 1) = u,
\]

\[
(0, -1) = v + w.
\]

If the dot product of a vector \(z = (z_1, z_2)\) with \(u, v\) and \(w\) is \(\leq 0\) then the dot product with each of the four vectors above is \(\leq 0\) so that \(z_1 = (1, 0) \cdot z \leq 0, -z_1 = (-1, 0) \cdot z \leq 0, z_2 = (0, 1) \cdot z \leq 0,\) and \(-z_2 = (0, -1) \cdot z \leq 0\) hence \(z_1 = z_2 = 0\) and therefore \(z = 0\).

**Problem 3.** For a triangle \(T_1\) construct a sequence of triangulations \(T_n\) where \(T_{n+1}\) is obtained by connecting the midpoints of each side of each triangle in \(T_n\) so that \(T_{n+1}\) has four times as many triangles as \(T_n\). Prove that the diameters of the triangles in \(T_n\) converge uniformly to zero.

**Answer** Use the fact that if two triangles with vertices \(ABC\) and \(DEF\) have the same angle at vertex \(B\) and vertex \(E\) and the ratios of sides are equal \(\frac{\text{len}(BA)}{\text{len}(BC)} = \frac{\text{len}(ED)}{\text{len}(EF)}\), then the triangles are similar and therefore the sides of all the triangulated triangles are half the lengths of the original triangle so that their diameters are half of the diameter of the original triangle. Therefore, the diameters converge uniformly to zero.

**Problem 4.** Do exercises 4, 5, 6, 7 on page 43.
**Answer** Exercise 4. Because $f$ maps the triangle into itself, for points $p$ on the boundary of the triangle the displacement vectors $v(p) := f(p) - p$ point into the triangle so the labels along a side with vertices labelled $A$ and $B$ must also be labelled $A$ or $B$ but never labelled $C$, and the same argument shows that the labels along the other two sides satisfies the criteria to be a Sperner labelling. Exercise 5. Form a sequence of divisions of the original tetrahedron into smaller and smaller tetrahera and form an infinite sequence of complete tetrahera (labelled according to the directions of the displacement vector field) and choose a subsequence of tetrahera whose centers converge to a point $p$ then show that the displacement vector field vanishes at $p$ and hence that $p$ is a fixed point. Exercise 6. The topological lemmas on pages 42 and 43 are contrapositive statements and are therefore logically equivalent. Exercise 7. A triangle is a cell because it is homeomorphic to a closed disk as show by constructing a homeomorphism. One construction is obtained by placing a disk inside the triangle then stretching along each radial arc of the disk so as to map its circular boundary onto the boundary of the triangle.