USC3002 Picturing the World Through Mathematics Test 1 Sem 2/2004

Problem 1. Let $D \subset \mathbb{R}^2$ be a closed disk and let $D_1$ and $D_2$ be two closed disks that satisfy $D_1 \subset D$, $D_2 \subset D$, and $D_1 \cap D_2 = \emptyset$. This means that $D_1$ and $D_2$ are subsets of $D$ and that they are disjoint. Consider the surface

$$S := D \setminus (D_1 \cup D_2),$$

formed by removing the disks $D_1$ and $D_2$ from $D$. Answer the following questions:

- (7 marks) Draw a picture of the set $S$.
- (7 Marks) Describe what the boundary of $S$ is?
- (8 Marks) Describe a partition of $S$ into disjoint subsets each of which is homeomorphic to exactly one of the following: (i) an open disk, (ii) an open interval of real numbers, (iii) a set containing a single point.
- (8 Marks) Use this partition to compute the Euler characteristic of $S$.

Problem 2. Construct homeomorphisms between:

- (7 marks) The interval $(0, 1]$ and the interval $[0, \infty)$,
- (7 Marks) The subsets

  \[ \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\} \text{ and } \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2\}, \]

- (8 Marks) The upper hemisphere

  \[ \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, 0 \leq z\} \]

  and the closed disk

  \[ \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}, \]

- (8 Marks) The interval $(0, 1)$ and the hyperbola

  \[ \{(x, y) \in \mathbb{R}^2 : 0 < x, xy = 1\}. \]
Problem 3. Write a few paragraphs explaining EXACTLY TWO (2) of the following concepts:

- (20 Marks) The Euler characteristic of a surface. Your discussion should explain what a partition is, give several examples of partitions and use them to compute Euler characteristics, and prove that the Euler characteristic computed from any two partitions is the same.

- (20 marks) The classification theorem for compact surfaces without boundary. Your discussion should explain how all such surfaces can be built from simple surfaces using connected sums, describe their topological properties, and give several examples.

- (20 Marks) The identification method for constructing topological spaces. Your discussion should explain both the general method and describe several specific constructions of topological spaces that include simple surfaces.

- (20 Marks) Continuity and uniform continuity of functions. Your discussion should provide definitions, give several examples that illustrate the difference between these concepts.