Question 1. (25 Marks) Consider a mechanical system that consists of a spring with stiffness constant $k$ such that one end is fixed and the other end is attached to a small (point like) object with mass $m$. The spring and object can only move along a line. Assume that the line is parameterized by the real variable $x$ such that $x = 0$ is the coordinate of the fixed end of the spring and $x = L$ is the coordinate of the equilibrium position of the object. Answer the following questions:

- (5 marks) Draw a picture of this system to show the relationship between the position $x(t)$ and the displacement $u(t)$ from equilibrium of the particle.

- (5 Marks) Derive the equation of motion for a small displacement. This equation will be a second order differential equation. Give the name of each physical law that you use in your derivation.

- (5 Marks) Describe the general solution of the equation of motion. The general solution will involve two parameters.

- (5 Marks) Show how to convert the equation of motion into a system of two first order differential equations.

- (5 Marks) Compute the smallest positive number $T$ such that for every $t$, if $u(t) = 0$ then $u(t + T) = 0$. The number $T$ is a function of $k$ and $m$.

Problem 2. (30 Marks) Consider the matrix

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

and the matrix-valued function

\[
W(t) = e^{At}, \quad t \in R.
\]

Compute each entry of $W(t)$ in terms of trigonometric functions using each of the following methods:
1. (10 marks) Directly by using the Taylor series and computing each term in the Taylor series and adding the terms.

2. (10 Marks) Express \( A = EDE^{-1} \) where \( D \) is a diagonal matrix and \( E \) is a matrix whose columns are eigenvectors of \( A \). Then express \( W(t) \) in terms of \( E \) and \( \exp(Dt) \) and complete the computation.

3. (10 Marks) Derive a differential equation and an initial condition for \( W(t) \) then derive differential equations and initial conditions for each entry in \( W(t) \) and solve these to compute each entry in \( W(t) \).

**Problem 3.** (15 Marks) Assume that \( M \) is a 2 by 2 matrix with eigenvectors \( v_1 \) and \( v_2 \) and \( M v_1 = -4 v_1 \) and \( M v_2 = -9 v_2 \). Compute the normal modes of a system that is described by the vibration equation

\[
\frac{d^2u}{dt^2} = Mu
\]

**Problem 4.** (15 Marks) Compute the value of the positive constant \( p \) that makes the equation of motion below critically damped and derive the specific solution for the critically damped equation if \( u(0) = 0 \) and \( \frac{du}{dt}(0) = 1 \).

\[
\frac{d^2u}{dt^2} + p \frac{du}{dt} + u = 0.
\]

**Problem 5.** (15 Marks) Compute the general solution and the value of \( \omega \) at the resonant frequency of the following equation:

\[
\frac{d^2u}{dt^2} + u = \cos \omega t.
\]