Cuts closed under a specified family of functions

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Main theme

In a model of arithmetic, control under which functions an initial segment is closed.

Plan of talk

- Nonstandard models of arithmetic
- Closing under a function
- Avoiding closing under a function
- Reflections
Nonstandard arithmetic

- The **language for arithmetic** is \( \{0, 1, +, \times, <\} \).
- **Peano arithmetic (PA)** consists of the axioms for *discretely ordered semirings*, and the *induction axiom*

\[
\theta(0) \land \forall x (\theta(x) \rightarrow \theta(x + 1)) \rightarrow \forall x \theta(x)
\]

for each formula \( \theta(x) \).
- A **nonstandard** model of PA is a model not isomorphic to \( \omega \).
- Skolem (1934) showed that nonstandard models exist.
Fix a nonstandard model $M \models \text{PA}$.

**Definition**
A *cut* of $M$ is a nonempty proper initial segment of $M$ that is closed under $x \mapsto x + 1$.

**Example**
$\omega$ is the *standard cut* of $M$.

**Definition**
An element $a \in M$ is *nonstandard* if $a > \omega$. 
Closing under definable functions

Definition
A function \( F : M \to M \) is \textit{definable} if
there is a formula \( \chi(x, y, z) \) and \( c \in M \) such that
\[
F(x) = y \iff M \models \chi(x, y, c)
\]
for all \( x, y \in M \).

Problem
How do we make cuts that are closed under a definable function \( F \)?

Assumption
All our functions \( F \) satisfy \( x \leq F(x) \leq F(x + 1) \) for all \( x \in M \).
Constructing a cut that is closed under $\times$

- Pick any nonstandard $a \in M$.
- Let $a^\omega = \sup\{a^k : k \in \omega\}$.
- Then $a^\omega$ is closed under $\times$.
- Notice $a^\omega$ is *not* closed under $x \mapsto x^x$. 
Primitive recursive functions

Definition (Grzegorczyk)
Set
\[ G_0(x) = x + 1, \]
\[ G_{k+1}(x) = G_k^{(x)}(x) \quad \text{for all} \; k, x. \]

Fact
There is a formula \( \chi(k, x, y) \) representing \( G_k(x) = y \).

Constructing a cut that is closed under the \( G_k \)'s

- Pick any nonstandard \( a \in M \).
- Let \( G_\omega(a) = \sup\{G_k(a) : k \in \omega\} \).
- Then \( G_\omega(a) \) is closed under \( G_k \) for all \( k \in \omega \).
- Notice \( G_\omega(a) \) is not closed under \( x \mapsto G_x(x) \).
Question (informal)
Is $G_\omega(a)$ closed under any function “other than” the $G_k$’s?

Definition
Let $F, G : M \to M$ and $I$ be a cut. Then $F$ dominates $G$ on $I$ if $F(x) \geq G(x)$ for all large enough $x \in I$.

Question (formal)
Is there a definable $F : M \to M$ under which $G_\omega(a)$ is closed that dominates $G_k$ on $G_\omega(a)$ for every $k \in \omega$?

Answer
Yes!
Diagonalization

\[ F(x) = G_{d(x)}(x) \]

\[ \begin{align*}
G_0(a) & \\
G_1(a) & \\
G_2(a) & \\
G_3(a) & \\
G_4(a) & \\
\cdots & \\
\end{align*} \]
What we know about $I = G_\omega(a)$

Summary

- $I$ is the smallest cut that contains $a$ and is closed under $G_k$ for all $k \in \omega$.
- There exists a definable function $F: M \to M$ under which $I$ is closed but dominates $G_k$ on $I$ for every $k \in \omega$.

Fact

The following are equivalent for a definable function $F: M \to M$.

- $I$ is closed under $F$.
- $F$ is dominated on $I$ by $x \mapsto G_d(x)(x)$ for some definable function $d: M \to M$ that satisfies $d(I) = \omega$. 
Closing exclusively

Definition
A cut $I$ is closed *exclusively* under the $G_k$'s if
- $I$ is closed under $G_k$ for every $k \in \omega$, and
- every definable function under which $I$ is closed is dominated by $G_k$ on $I$ for some $k \in \omega$.

Question
Are there cuts that are closed exclusively under the $G_k$'s?

Answer
Yes, at least when $M$ is countable.
A cut $I$ that is closed exclusively under the $G_k$’s

Search $I$ in a countable nonstandard $M \models \text{PA}$

Consider a definable $F : M \rightarrow M$.

- Suppose $I$ is to live between $a, b \in M$.
- We need $a \ll b$, i.e., $G_k(a) < b$ for all $k \in \omega$.

(a) Suppose $u \ll F(u)$ for some $u \in [a, b]$.

Then let $I$ live between such $u$ and $F(u)$.

(b) Suppose $u \not\ll F(u)$ for all $u \in [a, b]$.

Then $\max \{(\min k)(G_k(u) > F(u)) : u \in [a, b]\} \in \omega$.

So $F$ is dominated by $G_k$ on $I$ for some $k \in \omega$.

Repeat with another definable function $F'$ inside $[a', b']$.

Theorem

Every countable nonstandard model of PA contains continuum many cuts that are closed exclusively under the $G_k$’s.
Existentially closed models

- Existentially closed models are a counterpart of algebraically closed fields in model theory.
- They are models that satisfy a maximal number of $\exists$ formulas.
- “A cut $I$ not being closed under a function $F$” is existential:

  $$\exists x \in I \ \exists y > I \ y = F(x).$$

**Theorem**

A cut $I$ of $M$ is closed exclusively under the $G_k$’s if and only if $(M, I)$ is an existentially closed model of the theory

$$\text{PA} + \{I \text{ is a cut}\}$$
$$+ \{\forall x \in I \ \exists y \in I \ y = G_k(x) : k \in \omega\}.$$
Recall \( a \ll b \) means \( G_k(a) < b \) for all \( k \in \omega \).

We interpret \( a \ll b \) as “[\( a, b \] is large”.

Let \( I \) be a cut that is closed exclusively under the \( G_k \)’s.

Then \( M \) is homogeneous at \( I \), in the sense that every formula \( \theta \) that is satisfied arbitrarily close to \( I \) is satisfied densely in a neighbourhood of \( I \) with respect to \( \ll \).

**Question**

Can the model \( M \) be homogeneous in a larger region?

**Answer (Kaye, W)**

Yes, when \( M \) is countable arithmetically saturated.
Conclusion

What we saw

- There is a smallest cut $G_\omega(a)$ that contains a given $a \in M$ and is closed under a given definable family $(G_k)_{k \in \omega}$ of functions.
- This $G_\omega(a)$ is not closed exclusively under the $G_k$’s.
- There exist cuts that are closed exclusively under the $G_k$’s.
- This property is equivalent to being existentially closed.

Future work

- Automorphism group
- Independence results