Understanding $\text{BS}_{\Sigma_1} + \text{exp}$ via $\text{WKL}_0^*$

Tin Lok Wong

Kurt Gödel Research Center for Mathematical Logic
Vienna, Austria

Joint work with Ali Enayat (Gothenburg)

18 June, 2014

Financial support from FWF Project P24654-N25 is acknowledged.
Conservation Theorem (SSDP)

The following are equivalent for $M \models \text{PA}^\dagger$.

(a) $M \models \text{B} \Sigma_1 + \text{exp}.$

(b) $M$ expands to $(M, \mathcal{X}) \models \text{WKL}_0^\ast$.

Plan

1. Introduction
2. Arithmetized-Completeness-Theorem constructions
3. Arithmetized-Completeness-Theorem constructions iterated
4. Conclusion
$\text{BS}_{1} + \text{exp}$ and $\text{WKL}_0^*$

**Conservation Theorem (SSDP)**

The following are equivalent for $M \models \text{PA}^-$. 

(a) $M \models \text{BS}_{1} + \text{exp}$.

(b) $M$ expands to $(M, X) \models \text{WKL}_0^*$.

**BS$_1$** consists of $\text{PA}^-$ and
- $\Delta_0$-induction ($\text{I}\Delta_0$);
- $\Sigma_1$-collection.

**WKL$_0^*$** consists of extensionality, $\text{PA}^-$, and
- $\Sigma_0^0$-induction ($\text{I}\Sigma_0^0$);
- totality of exponentiation (exp);
- $\Delta_0^1$-comprehension ($\text{I}\Sigma_0^1$-CA); and
- Weak König Lemma (WKL), i.e., every unbounded binary tree has an unbounded path.

\[ \{ \text{RCA}_0^* \} \]

1st order arithmetic

2nd order arithmetic
The Conservation Theorem

Conservation Theorem (SSDP)
The following are equivalent for $M \models PA$.

(a) $M \models B\Sigma_1 + \exp$.
(b) $M$ expands to $(M, \mathcal{X}) \models WKL_0^*$.

Proof

(Simpson–Smith 1986) $WKL_0^* \vdash B\Sigma_1 + \exp$.

(Folklore) If $M \models I\Delta_0 + \exp$ has a proper end extension $K \models I\Delta_0$, then $(M, SSy_M(K)) \models WKL_0^*$.

(Dimitracopoulos–Paschalis) Every $M \models B\Sigma_1 + \exp$ has a proper end extension $K \models I\Delta_0$.

Remark
Simpson and Smith proved this for countable $M$ using forcing.
Arithmetized Completeness Theorem

Theorem (Simpson)
The following are equivalent over RCA\(^*\).
(a) WKL.
(b) Gödel’s Completeness Theorem, i.e., every consistent theory in first-order logic has a model.

Arithmetized Completeness Theorem for WKL\(^*\)
Let \((M, \mathcal{X}) \models WKL_{0}^*\) and \(T \supseteq \text{PA}^-\) be a theory coded in \((M, \mathcal{X})\).
If \((M, \mathcal{X}) \models \text{Con}(T)\), then there is \(K \models T\) end extending \(M\).

Theorem (Dimitracopoulos–Paschalis)
Every \(M \models B\Sigma_1 + \exp\) has a proper end extension \(K \models I\Delta_0\).

Theorem (Wilkie–Paris 1987)
\(B\Sigma_1 + \exp \vdash \text{TabCon}(I\Delta_0^*)\).
Cuts in models of Peano

Theorem (Kirby–Paris 1978, Enayat–W)
Let $M \models \text{I}_\Delta_0 + \exp$ that is short $\Pi_1$-recursively saturated. Then the following are equivalent.

(a) $M \models \text{B}_\Sigma_1 + \Pi_1 - \text{Th(PA)}$.
(b) $M$ has a proper end extension $K \models \text{PA}$.

Proof (essentially in Mc Aloon 1978)

- Expand $M$ to $(M, \mathcal{X}) \models \text{WKL}_0^*$.
- Reflection implies $\text{PA} \vdash \text{Con(I}_\Sigma_n)$ for every $n \in \mathbb{N}$.
- So $M \models \text{Con(I}_\Sigma_n)$ for every $n \in \mathbb{N}$.
- Overspill implies $M \models \text{Con(I}_\Sigma_\nu)$ for some $\nu > \mathbb{N}$.
- There is $K \models \text{I}_\Sigma_\nu \supseteq \text{PA}$ end extending $M$. □
Coded sets

Theorem (Dimitracopoulos–Paschalis), rephrased
The following are equivalent for $M \models I\Delta_0 + \text{exp}$.
(a) $M \models B\Sigma_1$.
(b) $M$ has a proper end extension $K \models I\Delta_0$.

Theorem (Enayat–W)
The following are equivalent for a countable $(M, \mathcal{X}) \models \text{RCA}^*_0$.
(a) $(M, \mathcal{X}) \models \text{WKL}^*_0$.
(b) $M$ has a proper end extension $K \models I\Delta_0$ where $\text{SSy}_M(K) = \mathcal{X}$.

Proof
A standard overspill argument shows (b) $\Rightarrow$ (a).
End extending $M$ to $K \models \text{I}\Delta_0$ with $\text{SSy}_M(K) = \mathcal{X}$

$M \models \text{WKL}^*_0 + \text{TabCon}(\text{I}\Delta^*_0)$

where $\mathcal{X} = \{D_n : n \in \mathbb{N}\}$

$(M, \mathcal{X}) \models \text{TabCon}(\text{ElemDiag}(K_n) + \text{\textquotedblleft}d_n \text{ codes } D_n\text{\textquotedblright})$
Completions of PA

Theorem (Kirby–Paris 1978, Enayat–W)
Let $M \models \mathrm{I} \Delta_0 + \exp$ that is short $\Pi_1$-recursively saturated. Then the following are equivalent.

(a) $M \models \mathrm{B} \Sigma_1 + \Pi_1\text{-Th(PA)}$.

(b) $M$ has a proper end extension $K \models \mathrm{PA}$.

Theorem (Wilkie 1977, Enayat–W)
Let $M \models \mathrm{I} \Delta_0 + \exp$ that is short $\Pi_1$-recursively saturated. The following are equivalent for a complete consistent $T \supseteq \mathrm{PA}$.

(a) $M \models \mathrm{B} \Sigma_1 + (\Pi_1 \cap T)$ and $(\Pi_n \cap T) \in \mathrm{SSy}(M)$ for all $n \in \mathbb{N}$.

(b) $M$ has a proper end extension $K \models T$.

Proof
(b) $\Rightarrow$ (a) is a simple application of the Sat$_\Pi_n$’s.
End extending \( M \) to \( K \models T \) following Lessan and Schmerl

\[
M \subseteq e K_n \models \text{Con}(\Pi_n^- \text{Diag}(\cdot) + (\Pi_{n+1} \cap T)) \in \Pi_{n+1} \cap T
\]

\[
T \supseteq \text{PA}
\]

\[
(\Pi_n \cap T) \in \text{SSy}(M)
\]

\[
(M, \mathcal{X}) \models \text{WKL}^* + (\Pi_1 \cap T)
\]

\[
K = \bigcup_{n \in \mathbb{N}} K_n \models T
\]
Further applications

Theorem (Enayat–W)

Let $M \models \mathrm{B} \Sigma_1 + \exp$ be nonstandard and short $(\Sigma_1 \cup \Pi_1)$-rec. sat.

The following are equivalent for a complete consistent $T \supseteq \mathrm{PA}$.

(a) $M \models \Pi_2 \cap T$ and $(\Pi_n \cap T) \in \mathrm{SSy}(M)$ for all $n \in \mathbb{N}$.

(b) $M$ has arbitrarily large initial segments $I \models T$.

Theorem (Schmerl 2014)

The following are equivalent for a countable $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$.

(a) $(M, \mathcal{X}) \models \mathrm{WKL}_0^*$ and $\mathcal{X} \supseteq \mathrm{Def}(M)$.

(b) $\mathcal{X} = \mathrm{SSy}_M(K)$ for some $K \supsetneq e M$.

parametrically definable sets
Finding \( K \supseteq_e M \) with \( \text{SSy}_M(K) = \mathcal{X} \) following Kaufmann

\[
\begin{align*}
(M, \mathcal{X}) & \models \text{WKL}_0^* + \text{PA} \\
\text{with } \mathcal{X} & = \{ D_n : n \in \mathbb{N} \} \supseteq \text{Def}(M)
\end{align*}
\]

\[
K = \bigcup_{n \in \mathbb{N}} K_n
\]

\[
K_n \supseteq_{\Pi_{n+2}} M \models \text{Con}(\Pi_{n+1}-\text{Diag}(\cdot) + \Pi_{n+3}-\text{Diag}(M)) \in \Pi_{n+2}
\]
Summary

Conservation Theorem (SSDP)
The following are equivalent for $M \models \text{PA}^-$.
(a) $M \models B\Sigma_1 + \text{exp}$.
(b) $M$ expands to $(M, \mathcal{X}) \models WKL_0^*$.

is useful for
(1) iterating Arithmetized-Completeness-Theorem constructions;
(2) controlling subsets coded in end extensions.

Questions
(i) Can we still pass on to second-order arithmetic without exp?
(ii) How necessary is PA in our theorems/arguments?