Models of Weak König’s Lemma

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This talk

Weak König’s Lemma (WKL)
Every infinite 0–1 tree has an infinite branch.

Plan

1. Motivation
2. Self-embeddings
3. Set-extensions
4. Conclusion

models of WKL $\cong$ coded subsets in end extensions
First-order arithmetic

- $L_1 = \{0, 1, +, \times, <\}$.
- A quantifier is \textit{bounded} if it is of the form $\forall v < t$ or $\exists v < t$.
- An $L_1$-formula is $\Delta_0$ if all its quantifiers are bounded.
- $\Sigma_n = \{\exists \bar{v}_1 \forall \bar{v}_2 \cdots Q \bar{v}_n \theta(\bar{v}, \bar{x}) : \theta \in \Delta_0\}$.
- The dual is called $\Pi_n$.
- A formula is $\Delta_n$ if it is both $\Sigma_n$ and $\Pi_n$.
- $I\Sigma_n$ consists of some basic axioms ($PA^-$) and for every $\theta \in \Sigma_n$,
  \[ \theta(0) \land \forall x \ (\theta(x) \rightarrow \theta(x + 1)) \rightarrow \forall x \ \theta(x). \]
- $B\Sigma_{n+1}$ consists of the axioms of $I\Sigma_0$ and for every $\theta \in \Sigma_{n+1}$,
  \[ \forall a \ (\forall x < a \ \exists y \ \theta(x, y) \rightarrow \exists b \ \forall x < a \ \exists y < b \ \theta(x, y)). \]
- $\exp$ asserts the totality of $x \mapsto 2^x$.

\textbf{Theorem (Paris–Kirby 1978; Parsons 1970; Parikh 1971)}

$I\Sigma_{n+1} \vdash B\Sigma_{n+1} \vdash I\Sigma_n$ for all $n \in \mathbb{N}$; and $I\Sigma_1 \vdash \exp$ but $B\Sigma_1 \not\vdash \exp$. 

$I\Sigma_0 \models M$
Cuts and end extensions

\[ n \in \mathbb{N} \]

\( \text{Definition} \)

Let \( I, M \models I \Sigma_0 \). Say \( I \) is a cut of \( M \), or \( M \) is an end extension of \( I \), if \( I \subseteq M \) and

\[ \forall i \in I \ \forall m \in M \setminus I \quad i \leq m. \]

In this case, write \( I \subseteq_e M \).

- \( I \Sigma_n \) consists of some basic axioms (\( \text{PA}^- \)) and for every \( \theta \in \Sigma_n \),

\[ \theta(0) \land \forall x \ (\theta(x) \rightarrow \theta(x + 1)) \rightarrow \forall x \ \theta(x). \]

Proposition (folklore)

(1) \( \mathbb{N} \) is a cut of every model of \( \Sigma_0 \), called the standard cut.

(2) If \( M \not \models \mathbb{N} \) and \( M \models \Sigma_n \), then \( \mathbb{N} \) is not \( \Sigma_n \)-definable in \( M \).

\( M \) is nonstandard
Second-order arithmetic

- $\mathcal{L}_I = \{0, 1, +, \times, <, \in\}$ has a \textit{number sort} and a \textit{set sort}.
- A quantifier is \textit{bounded} if it is of the form $\forall v < t$ or $\exists v < t$.
- $\Delta^0_0, \Sigma^0_n, \Pi^0_n, \Delta^0_n$ are defined as in $\mathcal{L}_I$.
- Formulas in $\bigcup_{n \in \mathbb{N}} \Sigma^0_n$ are called \textit{arithmetical}.
- $\Delta^0_1$-CA stands for the $\Delta^0_1$-comprehension axiom.
- $\text{RCA}_0 = \text{I} \Sigma^0_1 + \Delta^0_1$-CA. \quad $\text{RCA}_0^* = \text{B} \Sigma^0_1 + \text{exp} + \Delta^0_1$-CA.
- $\text{WKL}_0 = \text{RCA}_0 + \text{WKL}$. \quad $\text{WKL}_0^* = \text{RCA}_0^* + \text{WKL}$.
- If $M \models \text{I} \Sigma^0_1$, then $(M, \Delta^1$-Def$(M)) \models \text{RCA}_0 + \neg \text{WKL}$.
- If $M \models \text{B} \Sigma^0_1 + \text{exp}$, then $(M, \Delta^1$-Def$(M)) \models \text{RCA}_0^* + \neg \text{WKL}$.
- If $M \models \text{PA} = \bigcup_{n \in \mathbb{N}} \text{I} \Sigma^0_n$, then $(M, \text{Def}(M)) \models \text{WKL}_0$.

\textbf{Theorem (Harrington 1977)}

If $\sigma = \forall X \varphi(X)$ where $\varphi$ is arithmetical, then

\[ \text{WKL}_0 \models \sigma \quad \Rightarrow \quad \text{RCA}_0 \models \sigma. \]
Coded sets

Let $M \subseteq_e K \models I\Sigma_0$.

- Say $c \in K$ codes $S \subseteq M$ if
  $$ S = \{ x \in M : \text{the } x\text{th prime divides } c \}. $$

- Denote by $\text{Cod}(K/M)$ the set of all $S \subseteq M$ coded in $K$.

Theorem (Scott 1962)
If $M \not\subseteq_e K \models I\Sigma_0$ and $M \models \text{exp}$, then $(M, \text{Cod}(K/M)) \models \text{WKL}^*_0$.

Theorem (Enayat–W)
The following are equivalent for a countable $(M, \mathcal{X}) \models I\Sigma_0^0 + \text{exp}$.
(a) $(M, \mathcal{X}) \models \text{WKL}^*_0$.
(b) $\mathcal{X} = \text{Cod}(K/M)$ for some $K \not\subseteq_e M$ satisfying $I\Sigma_0$. 
Self-embeddings (pointwise fixing an initial segment)

Theorem (H. Friedman 1973; Dimitracopoulos–Paris 1988)

For every countable nonstandard $M \models I\Sigma_1$, there exist $I \subsetneq_e M$ and an isomorphism $M \rightarrow I$.

Theorem (Ressayre 1987)

The following are equivalent for all countable $M \models I\Sigma_0$.

(a) $M \not\cong \mathbb{N}$ and $M \models I\Sigma_1$.

(b) For every $a \in M$, there exist $I \subsetneq_e M$ and an isomorphism $M \rightarrow I$ which fixes all $x < a$.

Theorem (Tanaka 1997)

The following are equivalent for all countable $(M, \mathcal{X}) \models I\Sigma_0^0$.

(a) $M \not\cong \mathbb{N}$ and $(M, \mathcal{X}) \models WKL_0$.

(b) For every $a \in M$, there exist $I \subsetneq_e M$ and an isomorphism $(M, \mathcal{X}) \rightarrow (I, \text{Cod}(M/I))$ which fixes all $x < a$. 
Self-embeddings

Proposition (folklore)
If $M \not\cong \mathbb{N}$ and $M \models I\Sigma_1$, then $\mathbb{N}$ is not $\Delta_0(\Sigma_1)$-definable in $M$.

Theorem (Dimitracopoulos–Paris 1988)
The following are equivalent for a countable $M \models I\Sigma_0 + \exp$.
(a) $M \cong I$ for some $I \subsetneq e M$.
(b) $M \models B\Sigma_1$ and $\mathbb{N}$ is not parameter-free $\Delta_0(\Sigma_1)$-definable in $M$.

Theorem (Enayat–W)
The following are equivalent for a countable $(M, \mathcal{K}) \models I\Sigma_0^0 + \exp$.
(a) $(M, \mathcal{K}) \cong (I, \text{Cod}(M/I))$ for some $I \subsetneq e M$.
(b) $(M, \mathcal{K}) \models \text{WKL}^*$ and $\mathbb{N}$ is not parameter-free $\Delta_0(\Sigma_1)$-definable in $M$. 

Note: Not related to $\mathcal{K}$.
Tanaka’s Conjecture

Theorem (Harrington 1977)

If $\sigma = \forall X \varphi(X)$ where $\varphi$ is arithmetical, then

$$WKL_0 \vdash \sigma \implies RCA_0 \vdash \sigma.$$
The model theory behind Tanaka’s Conjecture

**Theorem (Simpson–Tanaka–Yamazaki 2002)**

If \( \sigma = \forall X \exists ! Y \varphi(X, Y) \) where \( \varphi \) is arithmetical, then

\[
WKL_0 \vdash \sigma \implies RCA_0 \vdash \sigma.
\]

**Lemma (Harrington 1977)**

Every countable \((M, X) \models RCA_0\) can be extended to \((M, Y) \models WKL_0\).

**Lemma (Simpson–Tanaka–Yamazaki 2002)**

Every countable \((M, X) \models RCA_0\) can be extended to \((M, Y_1), (M, Y_2) \models WKL_0\) such that

(a) \( Y_1 \cap Y_2 = X \); and

(b) \((M, Y_1)\) and \((M, Y_2)\) satisfy the same formulas with parameters from \((M, X)\).
Models of WKL $\approx$ coded subsets in end extensions

- **Ressayre, Tanaka**: Having an isomorphism onto a proper cut fixing any given initial segment characterizes $I\Sigma_1$ and $WKL_0$.
- **Dimitracopoulos–Paris, Enayat–W**: Having an isomorphism onto a proper cut is a sign of saturation.
- **Simpson–Tanaka–Yamazaki**: Any countable $(M, \mathcal{X}) \models RCA_0$ can be extended to $(M, \mathcal{Y}_1), (M, \mathcal{Y}_2) \models WKL_0$ with minimal intersection such that the same formulas with parameters from $(M, \mathcal{X}')$ are satisfied in them.

**Questions**

1. Can every $(M, \mathcal{X}') \models RCA_0^*$ be extended to $(M, \mathcal{Y}) \models WKL_0^*$?
2. **Scott 1962**: Given $(M, \mathcal{X}) \models WKL_0$, can one always find $K \supsetneq_e M$ satisfying $I\Sigma_0$ such that $\text{Cod}(K/M) = \mathcal{X}$?
3. Can every countable $(M, \mathcal{X}') \models RCA_0^*$ be extended to $(M, \Delta^0_1\text{-Def}(M, A)) \models RCA_0^*$ for some $A \subseteq M$?