Brunnian Braids and the Homotopy Groups
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Let M be a manifold and let

$$F(M,n) = \{(x_1, x_2, \ldots, x_n) \in M \times M \times \ldots \times M \mid x_i \neq x_j \text{ for } i \neq j\}$$

be the *ordered configuration space*. The symmetric group S_n acts on $F(M,n)$ by permuting coordinates. The *unordered configuration space* $B(M,n)$ is defined to be the quotient of $F(M,n)$ by the action of S_n. The *braid group* $B_n(M)$ of n strings over M is defined to be the fundamental group of $B(M,n)$. When $M = D^2$ is the disk, then $B_n(D^2)$ is the classical Artin braid group.

An intuitive description of $B_n(M)$ is as follows. Let β be an element in $B_n(M)$ represented by a loop λ in $B(M,n)$. Since $F(M,n) \to B(M,n)$ is a covering, the loop λ admits a unique path lifting $\tilde{\lambda}$ to $F(M,n)$. More precisely, let $(q_1, q_2, \ldots, q_n)$ be a base point in $F(M,n)$, that is, $q_1, q_2, \ldots, q_n$ are n distinguished points in M. Then $\tilde{\lambda}$ is a path in $F(M,n)$ staring with $(q_1, q_2, \ldots, q_n)$ and ending with a permutation $(q_{l_1}, q_{l_2}, \ldots, q_{l_n})$. From the path $\tilde{\lambda}\colon [0,1] \to F(M,n)$, $t \mapsto (\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))$, we obtain n paths $\lambda_1, \ldots, \lambda_n\colon [0,1] \to M$ with the property that $\lambda_i(t) \neq \lambda_j(t)$ for $i \neq j$ and $0 \leq t \leq 1$. In other words, we obtain n strings $\lambda_1, \ldots, \lambda_n$ in the cylinder $M \times [0,1]$ from the points $q_1, \ldots, q_n$ to $q_{l_1}, \ldots, q_{l_n}$, respectively.

A braid $\beta \in B_n(M)$ is called *Brunnian* if β becomes trivial when we remove ANY one of its strings. Let $\mathrm{Br}_n(M)$ be the set of Brunnian braids of n strings over M. Then $\mathrm{Br}_n(M)$ is a group under composition of strings.

Let $j\colon D^2 \to S^2$ be the canonical inclusion of the disk into the sphere, that is, j is the homeomorphism of the disk onto the upper hemisphere. Then j induces a group homomorphism $j_*\colon \mathrm{Br}_n(D^2) \to \mathrm{Br}_n(S^2)$.

Problem: Determine the cokernel of $j_*\colon \mathrm{Br}_n(D^2) \to \mathrm{Br}_n(S^2)$. In other words, determine the Brunnian braids in the cylinder $S^2 \times I$ modulo the Brunnian braids in the sub-cylinder $D^2 \times I \subseteq S^2 \times I$.

Note. 1. The image of $j_*\colon \mathrm{Br}_n(D^2) \to \mathrm{Br}_n(S^2)$ is a normal subgroup.
2. If $n \geq 5$, the cokernel of $j_*\colon \mathrm{Br}_n(D^2) \to \mathrm{Br}_n(S^2)$ is isomorphic to the homotopy group $\pi_{n-1}(S^3)$. This explains why the answer of above problem will be extremely interesting in homotopy theory. This also explains that the above problem might be very hard.
3. The kernel of $j_*\colon \mathrm{Br}_n(D^2) \to \mathrm{Br}_n(S^2)$ is isomorphic to $\mathrm{Br}_{n+1}(S^2)$ for $n \geq 5$.