ARTIN'S BRAID GROUPS, FREE GROUPS, AND THE LOOP SPACE OF THE 2-SPHERE

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Abstract. The purpose of this article is to describe connections between the loop space of the 2-sphere, and Artin's braid groups. The current article exploits Lie algebras associated to Vassiliev invariants in work of T. Kohno [19, 20], and provides connections between these various topics.

Two consequences are as follows:

(1) the homotopy groups of spheres are identified as "natural" sub-quotients of free products of pure braid groups, and
(2) an axiomatization of certain simplicial groups arising from braid groups is shown to characterize the homotopy types of connected CW-complexes.

1. A tale of two groups plus one more

In 1924 E. Artin [1, 2] defined the n-th braid group B_n together with the n-th pure braid group P_n, the kernel of the natural map of B_n to Σ_n the symmetric group on n-letters. It is the purpose of this article to derive additional connections of these groups to homotopy theory, as well as some overlaps with algebraic, and topological properties of braid groups.

This article gives certain relationships between free groups on n generators F_n, and braid groups which serve as a bridge between different structures. These connections, at the interface of homotopy groups of spheres, braids, and homotopy links, admit a common thread given by a simplicial group.

Recall that a simplicial group Γ_* is a collection of groups

$$\Gamma_0, \Gamma_1, \cdots, \Gamma_n, \cdots$$

together with face operations

$$d_i : \Gamma_n \to \Gamma_{n-1},$$

and degeneracy operations

$$s_i : \Gamma_n \to \Gamma_{n+1},$$

for $0 \leq i \leq n$. These homomorphisms are required to satisfy the standard simplicial identities.

One example is Milnor's free group construction $F[K]$ for a pointed simplicial set K with base-point $*$ in degree zero. The simplicial group $F[K]$ in degree n is the free group generated by the n simplices K_n modulo the single relation that $s_0^n(*) = 1$.

In case K is reduced, that is K consists of a single point in degree zero, the geometric realization of $F[K]$ is homotopy equivalent to $\Omega\Sigma|K|$ [24]. Theorem 1.1 below addresses one property concerning the simplicial group given by $F[\Delta[1]]$ where $\Delta[1]$ is the simplicial 1-simplex. This theorem gives one connection between $F[\Delta[1]]$, and a simplicial group described next which arises naturally from Artin's braid groups.

Key words and phrases. braid groups, simplicial groups, homotopy groups homotopy string links.

*Partially supported by DARPA grant number 2006-06918-01.

†Partially supported by the Academic Research Fund of the National University of Singapore R146-000-101-112.

Artin's pure braid group P_{n+1} is generated by elements $A_{i,j}$ for $1 \leq i < j \leq n+1$ with relations given in [23]; these relations as well as in an alternative form in terms of commutators are stated below in section 2. There is a simplicial group which in degree n is given by $\Gamma_n = P_{n+1}$ which is elucidated in section 2, as well as in [10, 3]. The simplicial structure is given heuristically as follows. Face operations are given by deletion of a strand, while the degeneracies are gotten by "doubling" of a strand. This simplicial group itself is denoted AP_*. Thus the pure braid groups P_k assemble into a simplicial group.

Theorem 1.1. *The loop space (as simplicial groups) of the simplicial group* AP_*, $\Omega(\mathrm{AP}_*)$, *is isomorphic to* $F[\Delta[1]]$ *as a simplicial group.*

Consider the free group on n letters $F_n = F_n[y_1, \cdots, y_n]$ together with elements x_i for $1 \leq i \leq n$ in P_{n+1} given by the naive "cabling" pictured in Figure 1 below.

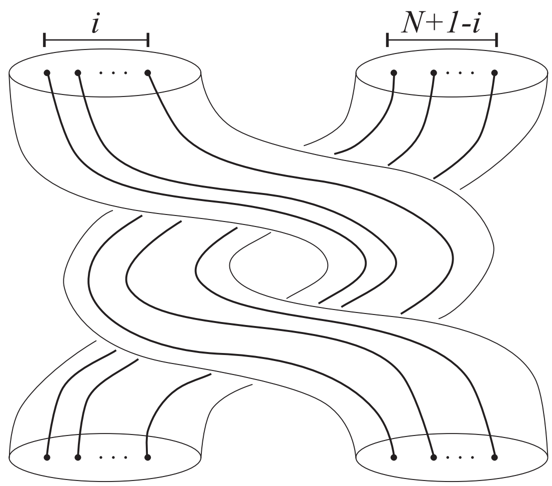

FIGURE 1. The braid x_i in P_{n+1}.

The braid x_1 with $N = 1 = i$ in Figure 1 is Artin's generator $A_{1,2}$ of P_2. The braids x_i for $1 \leq i \leq N$ in Figure 1 yield homomorphisms from a free group on N letters $F_N = F_N[y_1, \cdots, y_N]$ to P_{N+1}

$$\Theta_N \colon F_N[y_1, \cdots, y_N] \to P_{N+1}$$

defined on generators y_i in F_N by the formula

$$\Theta_N(y_i) = x_i.$$

A connection between Artin's braid group and the loop space of the 2-sphere is given next using the feature that the second pure braid group is isomorphic to the integers with a choice of generator denoted $A_{1,2}$. Notice that the simplicial circle S^1 has a single non-degenerate 1-simplex given by $\langle 0, 1\rangle$. Thus there exists an unique map of simplicial groups

$$\Theta : F[S^1] \to \ \mathrm{AP}_*$$

which in degree n is defined by

$$\Theta_n \colon F_n[y_1, \cdots, y_n] \to P_{n+1},$$

and specializes to

$$\Theta_1(\langle 0, 1\rangle) = A_{1,2}$$

with value given Artin's generator for P_2. One of the theorems stated in [10] is as follows.

Theorem 1.2. *The morphism of simplicial groups*

$$\Theta : F[S^1] \to \mathrm{AP}_*$$

is an embedding. Hence the homotopy groups of $F[S^1]$ *are natural sub-quotients of* AP_**, and the geometric realization of quotient simplicial set* $\mathrm{AP}_*/F[S^1]$ *is homotopy equivalent to the* 2*-sphere. Furthermore, the image of* Θ *is the smallest simplicial subgroup of* AP_* *which contains* $A_{1,2}$*.*

This theorem implies that the homotopy groups of $F[S^1]$, those of the loop space of the 2-sphere, are given as "natural" sub-quotients of the braid groups, a result related to work of the second author [28]. The proof of the above theorem sketched in [10] relies heavily on the structure of a Lie algebra arising from the "infinitesimal braid relations" as Vassiliev invariants of braids by work of T. Kohno [19, 20, 21], Falk, and Randell [15], as well as work of V. Drinfel'd [13] on the KZ equations. The precise details in section 8 here depend heavily on the specific structure of this Lie algebra.

In addition, the Lie algebraic methods used to prove this theorem suggest that the methods might be useful to study whether related maps are faithful. A "Lie algebraic/homological" criterion for testing whether a representation of P_n is faithful is given in [9].

The *Brunnian braid group* $Brunn_{n+1}$ is the subgroup of P_{n+1} given by those elements which are trivial after deleting any strand, that is the intersection of the kernels of all of the d_i, $0 \leq i \leq n$. Similarly, the *almost Brunnian braid group* $ABrunn_{n+2}$ is the subgroup of P_{n+2} given by those elements which are trivial after deleting any strand except possibly for the first strand, that is the intersecting of the kernels of the d_i for $1 \leq i \leq n+1$. Clearly, $d_0 : P_{n+2} \to P_{n+1}$, and $d_0|_{ABrunn_{n+2}} : ABrunn_{n+2} \to Brunn_{n+1}$ are surjections.

Corollary 1.3. *The cokernel of the natural map*

$$d_0|_{ABrunn_{n+2}} : ABrunn_{n+2} \cap F_{n+1} \to Brunn_{n+1} \cap F_n$$

is isomorphic to $\pi_{n+1}S^2$*.*

An example is presented next. The commutator of the braids $x_1 = A_{1,3} \cdot A_{2,3}$, and $x_2 = A_{1,2} \cdot A_{1,3}$ in the third braid group as presented below in section 6 represents the Hopf map $\eta : S^3 \to S^2$. In addition, the braid closure of the commutator $[x_1, x_2]$ gives the Borromean rings.

An analogue for all spheres arises at once by taking coproducts of simplicial groups $\mathrm{AP}_* \vee \mathrm{AP}_*$ which in degree n is given by the free product $P_{n+1} \amalg P_{n+1}$. Recall that if $n \geq 2$, the loop space of S^n is a retract of $\Omega(S^2 \vee S^2)$ by the Hilton-Milnor theorem.

Corollary 1.4. *The smallest simplicial subgroup of* $\mathrm{AP}_* \vee \mathrm{AP}_*$ *which contains* $P_2 \amalg P_2$ *in degree* 1 *is isomorphic to* $F[S^1] \vee F[S^1]$*. Hence* ΩS^n *is a retract, up to homotopy, of the geometric realization of the simplicial subgroup* $F[S^1] \vee F[S^1]$ *for any* $n \geq 2$*.*

The results in Corollary 1.4 suggest an axiomatization of certain families of simplicial groups. One application of Theorems 1.1, and 1.2 is given next. Let $\mathcal{B}$ denote the smallest full sub-category of the category of reduced simplicial groups which satisfies the following properties:

1. The simplicial group AP_* is in $\mathcal{B}$.
2. If Π, and Γ are in $\mathcal{B}$, then the coproduct $\Pi \vee \Gamma$ is in $\mathcal{B}$.
3. If Π is in $\mathcal{B}$, and Γ is a simplicial subgroup of Π, then Γ is in $\mathcal{B}$.
4. If Π is in $\mathcal{B}$, and Γ is a simplicial quotient group of Π, then Γ is in $\mathcal{B}$.

Theorem 1.5. *Let* X *denote simply-connected* CW*-complex. Then there exist an object* Γ_X *in* $\mathcal{B}$ *such that the loop space* $\Omega(X)$ *is homotopy equivalent to the geometric realization of* Γ_X*.*

The Lie algebra obtained from the descending central series of P_n, $E_0^*(P_n)$ occurs in several disparate ways in the results above. To close the introduction, these features are summarized next.

(1) One application of the structure for $E_0^*(P_n)$ is the proof of Theorem 1.2 which gives that the homomorphism $\Theta : F[S^1] \to \mathrm{AP}_*$ is an injection.
(2) A second application of $E_0^*(P_n)$ are the consequences of the axiomatization preceding Theorem 1.5.
(3) In the course of the proof of Theorem 1.2, the Lie algebra $E_0^*(P_n)$ is shown to naturally contain the E^0 term of the classical the Bousfield-Kan spectral sequence for the loop space of the 2-sphere. An analogous feature is satisfied for the classical unstable mod-p Adams spectral sequence for the loop space of the 2-sphere for which the descending central series is replaced by the mod-p descending central series of P_n as described in section 8.
(4) An important feature of $E_0^*(P_n)$ is the application in [19, 20] characterizing Vassiliev invariants of pure braids.
(5) Natural quotients of free groups, and P_n give a simplicial group $K[S^1]$ associated to natural self-maps of the loop space of double suspensions [7]. The same groups appear in work of Milnor on homotopy links [25, 26] as well as work of Habbeger-Lin [17] on homotopy string links.
(6) Related results concerning analogous structures for the pure braid group of the 2-sphere are given in [10, 3].
(7) A survey of how these topics 'fit' as well as other related topics is given in [10].

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The first author would like to thank the University of Tokyo, and the National University of Singapore for support during some of the work on this paper. The second author would like to thank the Université Montpellier II for support during some of the work on this project. Both authors would like to thank Volodia Vershinin for his suggestions.

2. The simplicial structure for AP_*

The pure braid group P_n is generated by elements $A_{r,s}$ for $1 \leq r < s \leq n$. The element $A_{r,s}$ may be thought of as linking the s-th strand around the r-th strand for $1 \leq r < s \leq n$ with a fixed orientation together with an explicit formula given by

$$A_{r,s} = \alpha(r,s) \cdot \sigma_r^2 \cdot \alpha(r,s)^{-1}$$

where

$$\alpha(r,s) = \sigma_{s-1} \cdot \sigma_{s-2} \cdot \sigma_{s-3} \cdots \sigma_{r+1}.$$

Artin gave a complete set of relations for the pure braid group P_n [2, 1], or [23]. A restatement of Artin's relations in terms of commutators is given next.

Theorem 2.1. *The group P_n is generated by elements*

$$A_{r,s}$$

for $1 \leq r < s \leq n$. A complete set of relations is given as follows:

(1) *If either $s < i$, or $k < r$, $A_{r,s}A_{i,k}A_{r,s}^{-1} = A_{i,k}$.*
(2) *If $i < k < s$, $A_{k,s}A_{i,k}A_{k,s}^{-1} = A_{i,s}^{-1}A_{i,k}A_{i,s}$.*
(3) *If $i < r < k$, $A_{r,s}A_{i,k}A_{r,s}^{-1} = A_{i,k}^{-1}A_{i,r}^{-1}A_{i,k}A_{i,r}A_{i,k}$.*
(4) *If $i < r < k < s$, $A_{r,s}A_{i,k}A_{r,s}^{-1} = A_{i,s}^{-1}A_{i,r}^{-1}A_{i,s}A_{i,r}A_{i,k}A_{i,r}^{-1}A_{i,s}^{-1}A_{i,r}A_{i,s}$.*

Furthermore, these relations are equivalent to the following relations.

(1) *If either $s < i$, or $k < r$, then $[A_{i,k}A_{r,s}] = 1$.*
(2) *If $i < k < s$, then $[A_{i,k}, A_{k,s}^{-1}] = [A_{i,k}, A_{i,s}]$.*
(3) *If $i < r < k$, $[A_{r,k}^{-1}, A_{i,k}^{-1}] = [A_{i,k}, A_{i,r}]$.*
(4) *If $i < r < k < s$, $[A_{i,k}, A_{r,s}^{-1}] = [A_{i,k}, [A_{i,r}, A_{i,s}]]$.*

The face operations in the simplicial group AP_* are defined as follows:

$$d_t(A_{i,j}) = \begin{cases} A_{i-1,j-1} & \text{if } t+1 < i, \\ 1 & \text{if t +1 = i}, \\ A_{i,j-1} & \text{if } i < t+1 < j\ , \\ 1 & \text{if t + 1 = j}, \\ A_{i,j} & \text{if } t+1 > i\ . \end{cases}$$

The degeneracy operations are defined as follows:

$$s_t(A_{i,j}) = \begin{cases} A_{i+1,j+1} & \text{if } t+1 < i\ , \\ A_{i,j+1} \cdot A_{i+1,j+1} & \text{if t+1 = i }, \\ A_{i,j+1} & \text{if } i < t+1 < j\ , \\ A_{i,j} \cdot A_{i,j+1} & \text{if t + 1 = j }, \\ A_{i,j} & \text{if } t+1 > j. \end{cases}$$

That the face and degeneracies preserve the defining relations is verified directly. The details are omitted. These operations give a convenient method for describing the behavior of Θ_n in the next section.

Recall that $F[S^1]$ in degree n is isomorphic to a free group on n letters $x_1, \ldots, x_n$. There are two natural choices for the definition of the face, and degeneracy operations. A short digression concerning natural topology, and these two different choices is given here.

The first description of the simplicial group structure arises by regarding the simplicial circle S^1 as a quotient of the simplicial 1-simplex with n-simplices given by

$$x_i = \langle 0, 0, \ldots, 0, 1, \ldots, 1 \rangle = \langle 0^i, 1^{n+1-i} \rangle$$

for $1 \leq i \leq n$.

$$d_i(x_j) = \begin{cases} x_{j-1} & \text{if } i < j, \text{ and} \\ x_j & \text{if } i \geq j. \end{cases}$$

The degeneracy operations are given by

$$s_i(x_j) = \begin{cases} x_{j+1} & \text{if } i < j, \text{ and} \\ x_j & \text{if } i \geq j. \end{cases}$$

A second natural description of the face, and degeneracy operations is obtained by applying a Tietze transformation as follows. Regard the free group on n-letters as the quotient

$$F[y_0, y_1, \ldots, y_n]$$

modulo the single defining relation

$$y_0 \cdot y_1 \cdots y_n = 1.$$

The face operations are then given by

$$d_i(y_j) = \begin{cases} y_j & \text{if } j < i \text{ ,} \\ 1 & \text{if i = j , and} \\ y_{j-1} & \text{if } j > i. \end{cases}$$

The degeneracy operations are given by

$$s_i(y_j) = \begin{cases} y_j & \text{if } j < i \text{ ,} \\ y_i \cdot y_{i+1} & \text{if i = j , and} \\ y_{j+1} & \text{if } j > i. \end{cases}$$

3. On looping AP_*

The next theorem was proven in [10] while a detailed proof is included here for completeness.

Theorem 3.1. *The loop space (as simplicial groups) of the simplicial group* AP_**,* $\Omega(\mathrm{AP}_*)$*, is isomorphic to* $F[\Delta[1]]$ *as a simplicial group. Thus* AP_* *is contractible, and the realization of the simplicial set* $\mathrm{AP}_*/F[S^1]$ *is homotopy equivalent to* S^2*.*

***Remark* 3.2.** *The long exact homotopy sequence of the fundamental fibration sequence due to Fadell, and Neuwirth* [14] *can be regarded as determining the homotopy type of the loop space for the simplicial group obtained from the pure braid groups.*

Proof. John Moore gave a definition for the loop space $\Omega\Gamma_*$ of a reduced simplicial group Γ_* (a simplicial group for which Γ_0 consists of a single element) [27]. This procedure corresponds to the topological notion of looping in the sense that the loop space of the geometric realization of Γ_* is homotopy equivalent to the geometric realization of $\Omega\Gamma_*$. This process of looping a simplicial group is described next.

Define a simplicial group $E\Gamma_*$ where the group $E\Gamma_n$ in degree n is given by the group Γ_{n+1} with face, and degeneracies given by "shifting down by one". Then $\Omega\Gamma_*$, the looping of Γ_* in degree n, is defined to be the kernel of the map

$$d_0 : E\Gamma_n \to \Gamma_n$$

thus

$$\Omega\Gamma_n = \ker[d_0 : \Gamma_{n+1} \to \Gamma_n].$$

In the special case that Γ_n is P_{n+1}, then $E\Gamma_n$ is P_{n+2}. Recall that P_{n+1} is generated by symbols $A_{i,j}$ for $1 \leq i < j \leq n+1$. Furthermore, the map $d_0 : E\Gamma_n \to \Gamma_n$ is induced by the projection map $p_* : \pi_1(F(\mathbb{R}^2, n+2)) \to \pi_1(F(\mathbb{R}^2, n+1))$ where $p : F(\mathbb{R}^2, n+2) \to F(\mathbb{R}^2, n+1)$ is the map given by projection to the first $n+1$ coordinates.

The fibre of the map p is $\mathbb{R}^2 - Q_{n+1}$ where Q_{n+1} denotes a set of cardinality $n+1$, and thus the kernel of p_* is isomorphic to F_{n+1}. It will be shown below that a choice of generators for this kernel, regarded as a simplicial set is $\Delta[1]$. Together with Theorem 1.2 giving that $\Theta : F[S^1] \to \mathrm{AP}_*$ is an embedding, the result follows.

The kernel of the map $d_0 : P_{n+2} \to P_{n+1}$ which is given by deleting the last coordinate is given by the free group with generators $P_{n+2,j}$ for $1 \leq j < n+2$. A basis for the kernel is given by $\{P_{n+2,j} | 1 \leq j \leq n+1\}$.

Furthermore, the simplicial 1-simplex is given by $\langle 0^i, 1^{n+1-i} \rangle$ in degree n with $0 \leq i \leq n+1$ with the single relation that $\langle 0^{n+1} \rangle = 1$ in the simplicial group $F[\Delta[1]$. There is a dimension-wise morphism of groups

$$\Psi : F[\Delta[1]] \to \Omega(\mathrm{AP}_*)$$

defined by sending $\langle 0^i, 1^{n+1-i} \rangle$ to $P_{n+2,i}$ for $0 < i < n+2$. A direct check gives that this dimension-wise homomorphism is compatible with the face, and degeneracies. Since each map is an isomorphism, the map Ψ is an isomorphism of simplicial groups. The theorem follows. □

4. On embeddings of residually nilpotent groups

Let $\rho : \pi \to G$ be a homomorphism between discrete groups. Let $\Gamma^n(\pi) = \Gamma^n$ denote the n-th stage of the descending central series for π. That is

(1) $\Gamma^1(\pi) = \pi$, and inductively
(2) $\Gamma^{n+1}(\pi) = [\Gamma^n, \pi]$, the subgroup generated by commutators

$$[\cdots [h_1, h_2]h_3] \cdots]h_q]$$

with h_i in π, and $q \geq n+1$.

Define the associated graded

$$E_0^n(\pi) = \Gamma^n(\pi)/\Gamma^{n+1}(\pi),$$

and

$$E_0^*(\pi) = \oplus_{n \geq 1} E_0^n(\pi).$$

It is a classical and easily checked fact that the commutator

$$[-,-] : \pi \times \pi \to \pi$$

given by

$$[x, y] = x^{-1} \cdot y^{-1} \cdot x \cdot y$$

induces the structure of Lie algebra on $E_0^*(\pi)$ [23]. A group homomorphism $\rho : \pi \to G$ preserves the stages of the descending central series. Thus there is an induced morphism of associated graded Lie algebras

$$E_0^*(\rho) : E_0^*(\pi) \to E_0^*(G).$$

Definition 4.1. A discrete group Γ is said to be residually nilpotent group if

$$\bigcap_{i \geq 1} \Gamma^i(\pi) = \{\text{identity}\}$$

where $\Gamma^i(\pi)$ denotes the i-th stage of the descending central series for π.

Theorem 4.2. *Assume that π is a residually nilpotent group. Let*

$$\rho : \pi \to G$$

be a homomorphism of discrete groups such that the morphism of associated graded Lie algebras

$$E_0^*(\rho) : E_0^*(\pi) \to E_0^*(G)$$

is a monomorphism. Then ρ is a monomorphism.

Proof. Let x denote a non-identity element in the kernel of ρ. Since π is residually nilpotent, there exists a natural number n such that the element x is in $\Gamma^n(\pi)$ and not in $\Gamma^{n+1}(\pi)$. But then x projects to a non-identity element in $E_0^n(\pi)$, and thus has non-trivial image in $E_0^n(G)$ contradicting the fact that x is a non-identity element in the kernel of ρ. □

If $F[S]$ is a free group generated by the set S, then $F[S]$ is residually nilpotent [23], and the next corollary follows at once.

Corollary 4.3. *If $F[S]$ is a free group generated by the set S, and*

$$E_0^*(\rho) : E_0^*(F[S]) \to \ E_0^*(G)$$

is a monomorphism of Lie algebras, then $\rho : F[S] \to \ G$ is a monomorphism of groups. If ρ is assumed to be an epimorphism, then it is an isomorphism.

Proof. If f induces an monomorphism on $E_0^*(F[S])$, then f is a monomorphism by the previous theorem. It suffices to note that f induces a surjection. The lemma follows. □

This approach for testing whether a homomorphism ρ is faithful is suited for residually nilpotent groups π such as free groups, and pure braid groups P_{n+1}. A single case is addressed here in section 8 here.

5. The Lie algebra associated to the descending central series for P_{n+1}

The next theorem gives the structure of the Lie algebra arising from the descending central series for P_k which was analyzed in work of T. Kohno [19, 20], Falk, and Randell [15], and Xicoténcatl [30]. Let $B_{i,j}$ denote the projections of the $A_{i,j}$ to $E_0^*(P_k)$.

Theorem 5.1. *The Lie algebra obtained from the descending central series for P_k is given by $\mathcal{L}_k$ the free Lie algebra generated by elements $B_{i,j}$ with $1 \leq i < j \leq k$, modulo the infinitesimal braid relations:*

- **(i):** $[B_{i,j}, B_{s,t}] = 0$ *if* $\{i,j\} \cap \{s,t\} = \phi$,
- **(ii):** $[B_{i,j}, B_{i,t} + B_{t,j}] = 0$ *if* $1 \leq i < t < j \leq k$, *and*
- **(iii):** $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ *if* $1 \leq i < t < j \leq k$.

Furthermore there is a split short exact sequence of Lie algebras

$$0 \to \ E_0^*(F_n) \xrightarrow{E_0^*(i)} E_0^*(P_{n+1}) \xrightarrow{E_0^*(d_n)} E_0^*(P_n) \to \ 0$$

where $E_0^*(F_n)$ is the free Lie algebra generated by $B_{i,n+1}$ for $1 \leq i < n+1$. In addition, $E_0^*(P_{n+1})$ is additively isomorphic to $E_0^*(P_n) \oplus E_0^*(F_n)$.

A related direct computation which is used below follows.

Proposition 5.2. (1) *If $1 \leq j < t < k \leq n+1$, then*

$$[B_{j,k} + B_{t,k}, B_{j,t}] = 0,$$

and

$$[B_{j,t}, B_{t,k}] = [B_{t,k}, B_{j,k}].$$

(2) *If $1 \leq r < m < n+1$, then*

$$[\sum_{1 \leq i \leq n} B_{i,n+1}, \sum_{1 \leq j \leq r} B_{j,m}] = 0.$$

Proof. Assume that $1 \leq j < t < k \leq n+1$, and consider the infinitesimal braid relations as follows:

- $[B_{j,t} + B_{j,k}, B_{t,k}] = 0$
- $[B_{j,t} + B_{t,k}, B_{j,k}] = 0$

Thus $[B_{j,k} + B_{t,k}, B_{j,t}] = [B_{t,k}, B_{j,k}] + [B_{j,k}, B_{t,k}] = 0$.

In addition, $[B_{j,t} + B_{j,k}, B_{t,k}] = 0$, and thus $[B_{j,t}, B_{t,k}] = [B_{t,k}, B_{j,k}]$.

Consider $[\sum_{1\leq i\leq n} B_{i,n+1}, B_{j,m}]$ for $m < n+1$. Since $[B_{i,n+1}, B_{j,m}] = 0$ if $\{i, n+1\} \cap \{j,m\} = \phi$, it follows that $[\sum_{1\leq i\leq n} B_{i,n+1}, B_{j,m}] = [B_{m,n+1} + B_{j,n+1}, B_{j,m}] = 0$ by the infinitesimal braid relations.

The proposition follows. □

6. On $\Theta_n : F_n \to P_{n+1}$

The proof that the map $\Theta_n : F_n \to P_{n+1}$ is a monomorphism depends on the structure of certain Lie algebras given in this section. These Lie algebras arise from passage to the associated graded for the descending central series filtration of a discrete group arising in the commutative diagram of groups

$$\begin{array}{ccc} F_n & \xrightarrow{\Theta_n} & P_{n+1} \\ \downarrow{\scriptstyle d_n} & & \downarrow{\scriptstyle d_n} \\ F_{n-1} & \xrightarrow{\Theta_{n-1}} & P_n. \end{array}$$

This diagram together with induction on n is used to prove the following theorem.

Theorem 6.1. *The maps $\Theta_n : F[x_1, x_2, \cdots, x_n] \to P_{n+1}$ on the level of associated graded Lie algebras*

$$E_0^*(\Theta_n) : E_0^*(F[x_1, x_2, \cdots, x_n]) \to E_0^*(P_{n+1})$$

are monomorphisms. Thus, by Theorem 4.2, the maps Θ_n are monomorphisms.

The hypothesis that Θ_1 is an isomorphism gives the initial step in an induction with the assumption that $E_0^*(\Theta_{n-1})$ is an embedding. To carry out the inductive step, notice that there is a commutative diagram of morphism of Lie algebras

$$\begin{array}{ccc} E_0^*(F_n) & \xrightarrow{E_0^*(\Theta_n)} & E_0^*(P_{n+1}) \\ \downarrow{\scriptstyle E_0^*(d_n)} & & \downarrow{\scriptstyle E_0^*(d_n)} \\ E_0^*(F_{n-1}) & \xrightarrow{E_0^*(\Theta_{n-1})} & E_0^*(P_n). \end{array}$$

Most of this section gives explicit results concerning these Lie algebras which are used to prove Theorem 6.1 in the next section. Thus recall P. Hall's classical result that

$$E_0^*(F_n) = E_0^*(F[x_1, x_2, \cdots, x_n])$$

is isomorphic to the free Lie algebra over the integers $\mathbb{Z}$

$$L[x_1, x_2, \cdots, x_n]$$

where the x_i's in the free Lie algebra are the projections of the analogous elements in the group F_n. Furthermore, the kernel of the projection map

$$\pi : L[x_1, x_2, \cdots, x_n] \to L[x_2, \cdots, x_n]$$

defined by

$$\pi(x_i) = \begin{cases} 0 & \text{if } i = 1, \\ x_i & \text{if } i > 1. \end{cases}$$

is given by $L[S_n]$ where S_n is the set

$$\{x_1, \mathrm{ad}(x_{i_1})\mathrm{ad}(x_{i_2}) \cdots \mathrm{ad}(x_{i_p})(x_1)\}$$

for all $p \geq 1$ with $i_j > 1$. Thus the Lie algebra kernel of

$$E_0^*(d_n) : E_0^*(F_n) \to E_0^*(F_{n-1})$$

is given by $L[S_n]$.

It is not the case that the exact sequence of groups $1 \to \ker(d_n) \to F_n \to F_{n-1} \to 1$ induces a short exact sequence of Lie algebras after passage to the sub-quotients of the descending central series. The details of this last assertion are not included.

It is convenient to give define additional elements specified in the following formulae.

(1) $\Lambda_n = B_{1,n+1} + B_{2,n+1} + \cdots + B_{n,n+1}$, and
(2) $\gamma_q(n) = -\Sigma_{n-q+2 \leq i \leq n} B_{i,n+1}$ if $2 \leq q \leq n$.

Some values of $\Theta(x_i)$ are required next, and are recorded in the next theorem which is a direct computation.

Theorem 6.2. *The map $\Theta_n : F[x_1, x_2, \cdots, x_n] \to P_{n+1}$ satisfies the following formula:*

$$E_0^*(\Theta_n)(x_q) = \Sigma_{1 \leq i \leq n-q+1 < j \leq n+1} B_{i,j}$$

on the level of associated graded Lie algebras $E_0^*(P_{n+1})$ for $1 \leq q \leq n$. In addition, the following formulas hold.

(1) $E_0^*(\Theta_n)(x_1) = \Lambda_n$.
(2) *If $j < n+1$, $[\Lambda_n, B_{i,j}] = 0$.*
(3) $E_0^*(\Theta_n)(x_q) = \Lambda_n + \gamma_q(n) + \Sigma_{1 \leq i \leq n-q+1 < j \leq n} B_{i,j}$.
(4) *If $2 \leq p \leq q$, then $[\gamma_p(n), \Theta(x_q(n))] = 0$.*

Proof. The formula

$$E_0^*(\Theta_n)(x_q) = \Sigma_{1 \leq i \leq n-q+1, n-q+2 \leq j \leq n} B_{i,j}$$

is that given by the degeneracies as described in section 6, the simplicial structure for AP_*. The next 4 formulae are proven next with formula 1 given by the definition of Λ_n.

With the condition $s < t < n+1$, notice that $[B_{i,n+1}, B_{s,t}] = 0$ for $i \neq s, t$. Thus

$$[\Lambda_n, B_{s,t}] = [\Sigma_{1 \leq i \leq n} B_{i,n+1}, B_{s,t}],$$

and

$$[\Lambda_n, B_{s,t}] = [B_{s,n+1} + B_{t,n+1}, B_{s,t}]$$

which is 0 by the infinitesimal braid relations in Proposition 5.2. Formula 2 follows.

Since

$$E_0^*(\Theta_n)(x_q) = \Sigma_{1 \leq i \leq n-q+1, n-q+2 \leq j \leq n} B_{i,j},$$

and

$$\Lambda_n + \gamma_q(n) = \Sigma_{1 \leq i \leq n-q+1} B_{i,n+1},$$

formula 3 that $E_0^*(\Theta_n)(x_q) = \Lambda_n + \gamma_q(n) + \Sigma_{1 \leq i \leq n-q+1 < j \leq n} B_{i,j}$ follows directly.

To work out part 4, consider

$$[\gamma_p(n), \Theta(x_q)] = -[\Sigma_{n-p+1 \leq i \leq n} B_{i,n+1}, \Sigma_{1 \leq i \leq n-q+1 < \leq j \leq n+1} B_{i,j}].$$

Then expand each term for $n - p + 1 \leq i \leq n$ given by

$$[B_{i,n+1}, \Sigma_{1 \leq i \leq n-q+1 < \leq j \leq n+1} B_{i,j}].$$

There are two cases to check.

(1) If $i < n-q+1$, then $[B_{i,n+1}, \Sigma_{1 \leq i \leq n-q+1 < \leq j \leq n+1} B_{i,j}] = 0$ by the infinitesimal braid relations.
(2) If $i = n-q+1$, then $[B_{n-q+1,n+1}, \Sigma_{1 \leq i \leq n-q+1 < \leq j \leq n+1} B_{i,j}] = 0$ by the infinitesimal braid relations.

The theorem follows. □

Some additional properties concerning the degeneracies are given next follow at once.

Theorem 6.3. *The map $\Theta_n : F[x_1, x_2, \cdots, x_n] \to P_{n+1}$ satisfies the following formula:*

(1) *For any fixed $0 \leq j \leq n$, $s_j \Lambda_n = \Lambda_{n+1}$.*
(2)
$$s_j(\gamma_q(n)) = \begin{cases} \gamma_q(n+1) & \text{if } j < n+1-q, \\ \gamma_{q+1}(n+1) & \text{if } j \geq n+1-q. \end{cases}$$
(3) $[\gamma_3(3), \Theta(x_2)] = [\gamma_3(3), \Lambda_3] + [\Lambda_3, \gamma_2(3)] + 2[\gamma_3(3), \gamma_2(3)]$.
(4) *For any fixed $2 \leq i < j \leq n$, there are sequences of degeneracies $s(I, J)$ such that*
- $s(I, J)(x_2) = x_i$,
- $s(I, J)(x_3) = x_j$,
- $s(I, J)([\gamma_3(3), E_0^*(\Theta_n)(x_2)] = [\gamma_j(n), E_0^*(\Theta_n)(x_i)]$.

(5) *For any fixed $2 \leq i < j \leq n$,*
$$[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)] + 2[\gamma_j(n), \gamma_i(n)].$$
(6) *If $2 \leq p \leq q$, then*
$$[\gamma_p(n), \Theta_n(x_q(n))] = 0.$$
(7) *If $2 \leq q < p$, then*
$$[\gamma_p(n), \Theta_n(x_q(n)) - 2\gamma_q(n)] = [\gamma_p(n), \Lambda_n] + [\Lambda_n, \gamma_q(n)].$$

Proof. The degeneracies are described in section 2 which gives the simplicial structure for AP_*. Recall that $\Lambda_n = \Sigma_{1 \leq i \leq n} B_{i,n+1}$. That $s_j \Lambda_n = \Lambda_{n+1}$, and formula 1 follows at once. Recall that $\gamma_q(n) = B_{n+2-q,n+1} + B_{n+3-q,n+1} + \cdots + B_{n,n+1}$. Thus if $j \leq n+1-q$, then $s_j \gamma_q(n) = B_{n+3-q,n+2} + B_{n+4-q,n+2} + \cdots + B_{n+1,n+2}$. Furthermore, if $n+2-q \leq j$, then $s_j \gamma_q(n) = B_{n+2-q,n+2} + B_{n+3-q,n+2} + \cdots + B_{n+1,n+2}$, and formula 2 follows.

To check formula 3, notice that the following hold.

(1) By definition, $\gamma_3(3) = -(B_{2,4} + B_{3,4})$.
(2) By definition, $E_0^*(\Theta_3)(x_2) = B_{1,3} + B_{2,3} + B_{1,4} + B_{2,4}$.
(3) Thus $[\gamma_3(3), E_0^*(\Theta_3)(x_2)] = -[B_{2,4} + B_{3,4}, B_{1,3} + B_{2,3} + B_{1,4} + B_{2,4}]$ which equals
(4) $-[B_{2,4}, B_{1,3} + B_{2,3} + B_{1,4} + B_{2,4}]$ by the infinitesimal braid relations.
(5) Thus $[\gamma_3(3), E_0^*(\Theta_3)(x_2)] = -[B_{2,4}, B_{2,3} + B_{1,4}]$ by the infinitesimal braid relations.
(6) Furthermore, $[B_{2,4}, B_{2,3} + B_{3,4}] = 0$ by the infinitesimal braid relations, and thus
(7) $[\gamma_3(3), \Theta(x_2)] = [B_{2,4}, B_{3,4}] - [B_{2,4}, B_{1,4}] = [B_{2,4}, B_{3,4}] - [B_{2,4}, \Lambda_3 - B_{3,4}]$.
(8) Substituting $\gamma_2(3) = -B_{3,4}$, $\gamma_3(3) = -(B_{2,4} + B_{3,4}$, and $\gamma_2(3) - \gamma_3(3) = B_{2,4}$ gives $[\gamma_3(3), \Theta(x_2)] = [\gamma_3(3), \Lambda_3] + [\Lambda_3, \gamma_2(3)] + 2[\gamma_3(3), \gamma_2(3)]$. Thus formula 3 follows.

Formula 4 clearly follows from the degeneracies.

To check formula 5, notice that
$$[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = s(I, J)([\gamma_3(3), E_0^*(\Theta_n)(x_2)],$$
$$s(I, J)([\gamma_3(3), E_0^*(\Theta_3)(x_2)] = S(I, J)([\gamma_3(3), \Lambda_3] + [\Lambda_3, \gamma_2(3)] + 2[\gamma_3(3), \gamma_2(3)]),$$
and
$$S(I, J)([\gamma_3(3), \Lambda_3] + [\Lambda_3, \gamma_2(3)] + 2[\gamma_3(3), \gamma_2(3)])$$
equal to
$$[\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)] + 2[\gamma_j(n), \gamma_i(n)].$$

Formula 6 is formula 4 in Theorem 6.2.

By formula 5, For any fixed $2 \leq i < j \leq n$,

$$[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)] + 2[\gamma_j(n), \gamma_i(n)].$$

Thus

$$[\gamma_j(n), E_0^*(\Theta_n)(x_i) - 2\gamma_i(n)] = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)],$$

and formula 7 follows.

□

7. On the proof of Theorem 6.1

Recall the morphism of Lie algebras

$$\begin{array}{ccc} E_0^*(F_n) & \xrightarrow{E_0^*(\Theta_n)} & E_0^*(P_{n+1}) \\ \downarrow{\scriptstyle E_0^*(d_n)} & & \downarrow{\scriptstyle E_0^*(d_n)} \\ E_0^*(F_{n-1}) & \xrightarrow{E_0^*(\Theta_{n-1})} & E_0^*(P_n). \end{array}$$

Taking Lie algebra kernels results in a commutative diagram

$$\begin{array}{ccc} L[x_1^{E_0^*(F_{n-1})}] & \xrightarrow{E_0^*(\Theta_n)} & L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}] \\ \downarrow & & \downarrow \\ E_0^*(F_n) & \xrightarrow{E_0^*(\Theta_n)} & E_0^*(P_{n+1}) \\ \downarrow{\scriptstyle E_0^*(d_n)} & & \downarrow{\scriptstyle E_0^*(d_n)} \\ E_0^*(F_{n-1}) & \xrightarrow{E_0^*(\Theta_{n-1})} & E_0^*(P_n) \end{array}$$

for which

- the Lie algebra kernel of $E_0^*(d_n) : E_0^*(F_n) \to E_0^*(F_{n-1})$ is $L[x_1^{E_0^*(F_{n-1})}]$, and
- the Lie algebra kernel of $E_0^*(d_n) : E_0^*(P_{n+1}) \to E_0^*(P_n)$ is

$$L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}].$$

The inductive hypothesis is that $E_0^*(\Theta_{n-1})$ is a monomorphism. To finish, it suffices to check that the induced map of Lie algebras

$$L[x_1^{E_0^*(F_{n-1})}] \xrightarrow{E_0^*(\Theta_n)} L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}]$$

is a monomorphism. This will be checked using the computations of section 5.

The next step is to consider the morphism of Lie algebras

$$p : L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}] \to L[B_{2,n+1}, \cdots, B_{n,n+1}]$$

defined by the formula

$$p(B_{j,n+1}) = \begin{cases} B_{j,n+1} & \text{if } j > 1, \\ -(B_{2,n+1} + B_{3,n+1} + \cdots + B_{n,n+1}) & \text{if } j = 1. \end{cases}$$

Notice that $p(\Sigma_{1 \leq j \leq n} B_{j,n+1}) = 0$, and so

$$p(\Lambda_n) = 0.$$

A remark is appropriate here. With the exception of the map p, all of the above maps are morphisms of simplicial Lie algebras. The map p fails to preserve one face operation, and is thus not a morphism of simplicial Lie algebras.

Let B equal $E_0^*(F_{n-1})$. The image of

$$E_0^*(\Theta_n) : L[x_1^B] \to L[B_{1,n+1}, B_{2,n+1}, \cdots, B_{n,n+1}]$$

is in the Lie ideal generated by $\Theta(x_1) = \Lambda_n$, and thus is in the Lie algebra kernel of p. Hence the map Θ restricts to a map of Lie algebras

$$E_0^*(\Theta_n) : L[x_1^B] \to \ L[\Lambda_n^C]$$

where $C = L[B_{2,n+1}, B_{3,n+1}, \cdots, B_{n,n+1}]$. In the work which is given below, the notation $x_j = x_j(n)$, and $\gamma_j = \gamma_j(n)$ is used when the integer n is clear from the context.

The structure of the Lie algebras $L[x_1^B]$, and $L[\Lambda_n^C]$ are used below. Recall that the abelianization of the Lie algebra $L[S]$ is equal to

$$H_1(L[S]) = L[S]/([L[S], L[S]]).$$

Some of this structure is well-known, and recorded next.

Theorem 7.1. *The abelianization of $L[x_1^B]$, $H_1L[x_1^B]$, is given by the free abelian group with basis x_1, and*

$$[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$$

for all sequences $2 \leq j_1, j_2, \cdots, j_q$.

The graded free abelian groups $H_1L[x_1^B]$, and $H_1L[\Lambda_n^C]$ are filtered as described next for which the filtration of $H_1L[\Lambda_n^C]$ satisfies a reversed ordering from that of $H_1L[x_1^B]$. This reversal is forced by the infinitesimal braid relations, and the requirement that the map $E_0^*(\Theta_n)$ preserve filtrations on the level of the first homology group as suggested by Theorems 7.4, and 7.7 below.

Definition 7.2. Grade the first homology group $H_1(L[x_1^B])$ by setting

$$Gr_q(H_1(L[x_1^B])$$

for $q \geq 1$ equal to the linear span of the classes

$$[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$$

for all sequences $2 \leq j_1, j_2, \cdots j_{q-1}, j_q$ with the convention that $Gr_1(H_1(L[x_1^B])$ is the linear span of of the class x_1. Thus

$$H_1(L[x_1^B]) = \oplus_{q\geq 1} Gr_q(H_1(L[x_1^B]).$$

Filter $Gr_q(H_1(L[x_1^B])$ by $F_p(Gr_q(H_1(L[x_1^B]))$ as follows.

(1) For every $q \geq 1$,

$$F_0Gr_q(H_1(L[x_1^B])$$

is the linear span of the classes

$$[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$$

for all sequences

$$2 \leq j_1 \leq j_2 \leq \cdots \leq j_q.$$

(2) Filtration $p > 0$, $F_p(Gr_q(H_1(L[x_1^B]))$, is the linear span of the classes

$$[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_q}]x_{j_q}]$$

with $D([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_q}]x_{j_q}]) \leq p$ where

$$D([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_q}]x_{j_q}]) = \Sigma_{1\leq i\leq q-1} d(x_{j_i})$$

with $d(x_{j_i})$ equal to the number of $x_{j_{i+k}}$ such that $j_i > j_{i+k}$ for $k > 0$.

Thus, there are inclusions

$$F_p(Gr_q(H_1(L[x_1^B])) \subset F_{p+1}(Gr_q(H_1(L[x_1^B]))$$

for all p, and

$$Gr_q(H_1(L[x_1^B])) = \cup_{0\leq p} F_p(Gr_q(H_1(L[x_1^B])).$$

There is a similar filtration $H_1L[\Lambda_n^C]$, but with reversed ordering.

Definition 7.3. Grade the first homology group

$$H_1(L[\Lambda_n^C])$$

by setting

$$Gr_q(H_1(L[\Lambda_n^C])$$

for $q \geq 1$ equal to the linear span of the classes

$$[\cdots[\Lambda_n, \gamma_{j_1}]\gamma_{j_2}]\cdots]\gamma_{j_{q-1}}]\gamma_{j_q}]$$

for all sequences $2 \leq j_1, j_2, \cdots j_{q-1}, j_q$ with the convention that $Gr_1(H_1(L[\Lambda_n^C])$ is the linear span of the class x_1. Thus

$$H_1(L[\Lambda_n^C]) = \oplus_{q\geq 1} Gr_q(H_1(L[\Lambda_n^C]).$$

Filter $Gr_q(H_1(L[\Lambda_n^C])$ by $F_p(Gr_q(H_1(L[\Lambda_n^C]))$ as follows.

(1) For every $q \geq 1$,

$$F_0 Gr_q(H_1(L[\Lambda_n^C])$$

is the linear span of the classes

$$[\cdots[\Lambda_n, \gamma_{j_1}]\gamma_{j_2}]\cdots]\gamma_{j_{q-1}}]\gamma_{j_q}]$$

for all sequences

$$j_1 \geq j_2 \geq \cdots \geq j_q \geq 2.$$

(2) Filtration $p > 0$, $F_p(Gr_q(H_1(L[\Lambda_n^C]))$ is the linear span of the classes

$$[\cdots[\Lambda_n, \gamma_{j_1}]\gamma_{j_2}]\cdots]\gamma_{j_{q-1}}]\gamma_{j_q}]$$

such that

$$\Delta([\cdots[\Lambda_n, \gamma_{j_1}]\gamma_{j_2}]\cdots]\gamma_{j_{q-1}}]\gamma_{j_q}]) = \Sigma_{1\leq i\leq q-1}\delta(\gamma_{j_i})$$

with $\delta(\gamma_{j_i})$ equal to the number of $\gamma_{j_{i+k}}$ such that $j_i < j_{i+k}$ for $k > 0$.

Thus, there are inclusions

$$F_p(Gr_q(H_1(L[\Lambda_n^C])) \subset F_{p+1}(Gr_q(H_1(L[\Lambda_n^C]))$$

for all p, and

$$Gr_q(H_1(L[\Lambda_n^C])) = \cup_{0\leq p} F_p(Gr_q(H_1(L[\Lambda_n^C])).$$

Theorem 7.4. *The map of Lie algebras $E_0^*(\Theta_n) : L[x_1^B] \to L[\Lambda_n^C]$ induces a map*

$$H_1(\Theta_n) : H_1L[x_1^B] \to H_1L[\Lambda_n^C]$$

which

(1) *preserves both gradation, and filtration as given in Definitions 7.2, and 7.3, and*
(2) *is an isomorphism.*

Corollary 7.5. *The map of Lie algebras $E_0^*(\Theta_n) : L[x_1^B] \to L[\Lambda_n^C]$ is an isomorphism.*

The proof of Theorem 7.4 follows by induction with the first case given next in which $x_i = x_i(n)$, and $\gamma_i = \gamma_i(n)$.

Theorem 7.6. *If $n \geq 1$, then $E_0^*(\Theta_n)(x_1) = \Lambda_n$. If $2 \leq j_1 \leq j_2 \leq \cdots \leq j_q$, then*

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]) = [\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}].$$

Thus $E_0^*(\Theta_n)$ preserves filtration 0 in Theorem 7.4, and induces an isomorphism

$$H_1(\Theta_n) : F_0(Gr_q(H_1(L[x_1^B])) \to F_0(Gr_q(H_1(L[\Lambda_1^C]))$$

for all $q \geq 0$.

Proof. By Theorem 6.2, $E_0^*(\Theta_n)(x_1) = \Lambda_n$. Thus the theorem is correct in the case of the empty sequence. The next step is to check that $E_0^*(\Theta_n)$ preserves filtration 0 by inducting on q as stated in Theorem 7.4. Assume by induction that if $2 \leq j_1 \leq j_2 \leq \cdots \leq j_q$, then

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]) = [\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}].$$

Next, consider $j_q \leq j_{q+1}$, together with the value of

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]x_{j_{q+1}}])$$

given by

$$[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}]E_0^*(\Theta_n)(x_{q+1})].$$

If $2 \leq j_p \leq j_q$, then $[\gamma_{j_p}(n), \Theta(x_{j_q}(n))] = 0$ by Theorem 6.2. Hence by the Jacobi identity, $[[A, B], C] = [[A, C], B] + [A, [B, C]]$, the value of

$$[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}]E_0^*(\Theta_n)(x_{q+1})]$$

is equal to

$$[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]E_0^*(\Theta_n)(x_{q+1})]\gamma_{j_1}].$$

Notice that by the inductive hypothesis, the element

$$[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]E_0^*(\Theta_n)(x_{q+1})]$$

is equal to

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_2}]x_{j_3}]\cdots]x_{j_{q-1}}]x_{j_{q+1}}]).$$

That is, the element x_{j_1} does not appear.

Furthermore, notice that the element $E_0^*(\Theta_n)([\cdots[x_1, x_{j_2}]x_{j_3}]\cdots]x_{j_{q-1}}]x_{j_{q+1}}])$ is equal to

$$[\cdots[\Lambda_n, \gamma_{j_{q+1}}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_3}]\gamma_{j_2}]$$

by the inductive hypothesis. Hence

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]x_{j_{q+1}}]) = [\cdots[\Lambda_n, \gamma_{j_{q+1}}]\gamma_{j_q}]\cdots]\gamma_{j_2}]\gamma_{j_1}].$$

The theorem follows. □

The proof of the next theorem is analogous, but uses a variation of Theorem 6.3 to measure changes of order, and their effect on the filtration defined above. To keep track of certain signs, the following conventions are used.

Theorem 7.7. *If $2 \leq j_1, j_2, \cdots, j_q$, then the class of*

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}])$$

in $H_1L[\Lambda_n^C]$ is equal to the class of

$$(\pm 1)[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}] + \Omega$$

where Ω projects to an element of lower filtration degree in $H_1L[\Lambda_n^C]$. In addition, $E_0^*(\Theta_n)$ both preserves the filtrations as given in Definitions 7.2, and 7.3 , as well induces an isomorphism on the level of first homology groups.

Proof. The proof of Theorem 7.7 is by induction on filtration degree p for fixed gradation $Gr_q(H_1(L[x_1^B])$ starting with filtration degree $p = 0$. Notice that if $n \geq 1$, then $E_0^*(\Theta_n)(x_1) = \Lambda_n$, and if $2 \leq j_1 \leq j_2 \leq \cdots \leq j_q$, then

$$E_0^*(\Theta_n)([\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]) = [\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}]$$

by 7.6. Thus the theorem is correct for filtration degree 0, and for all q as the map $E_0^*(\Theta_n)$ induces an isomorphism on the level of

$$H_1(\Theta_n) : F_0Gr_q(H_1(L[x_1^B])) \rightarrow\ F_0Gr_q(H_1(L[\Lambda_n^C]))$$

for all $1 \leq q$.

Thus assume inductively that

$$E_0^*(\Theta_n)[\cdots[x_1,x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$$

in $H_1L[\Lambda_n^C]$ is equal to $(\pm 1)[\cdots[\Lambda_n,\gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}]+\Omega$ where Ω is of lower filtration as stated in Theorem 7.7.

Consider the following formulae.

(1) $E_0^*(\Theta_n)[\cdots[x_1,x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]x_{j_{q+1}}]=[[A,B],C]$ for which

$$A=E_0^*(\Theta_n)([\cdots[x_1,x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}])$$

with

$$B=E_0^*(\Theta_n)(x_{j_q}),$$

and

$$C=E_0^*(\Theta_n)(x_{j_{q+1}}).$$

(2) The inductive hypothesis gives that

$$[A,B]=(\pm 1)[\cdots[\Lambda_n,\gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}]+\Omega$$

for which Ω projects to lower filtration degree in $Gr_q(H_1(L[x_1^B]))$. Hence

$$[[A,B],C]=[((\pm 1)[\cdots[\Lambda_n,\gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}]+\Omega)],C]$$

in $Gr_q(H_1(L[x_1^C]))$.

(3) Since Ω has lower filtration degree than $[A,B]$, the filtration degree of $[\Omega,C]$ has filtration degree strictly less than that of $[[A,B],C]$ in $Gr_q(H_1(L[\Lambda_n^C])$ by inspection of the definition.

(4) Next consider

$$[\cdots[\Lambda_n,\gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}],C]=[[E,F]C]=[[E,C]F]+[E[F,C]]$$

for which
- $E=[\cdots[\Lambda_n,\gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]$, and
- $F=\gamma_{j_{q-1}}$.

(5) Notice that

$$[E,C]=[\cdots[\Lambda_n,\gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]E_0^*(\Theta_n)(x_{j_{q+1}})].$$

(6) Thus $[E,C]$ is the sum

$$[[X,E_0^*(\Theta_n)(x_{j_{q+1}})]\gamma_{j_2}]+[X,[\gamma_{j_2},E_0^*(\Theta_n)(x_{j_{q+1}})]]$$

where

$$X=\cdots[\Lambda_n,\gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_3}].$$

(7) If

$$j_2\le j_{q+1},$$

then $[\gamma_{j_2},E_0^*(\Theta_n)(x_{j_{q+1}})]=0$ by part 6 of Theorem 6.3. Hence

$$[E,C]=[[X,E_0^*(\Theta_n)(x_{j_{q+1}})]\gamma_{j_2}],$$

the inductive hypothesis applies, and the Theorem follows.

(8) If

$$j_2>j_{q+1},$$

then

$$[\gamma_{j_2},E_0^*(\Theta_n)(x_{j_{q+1}})]=V+Y$$

where $V=[\gamma_j(2),\Lambda_n]+[\Lambda_n,\gamma_{j_{q+1}}]$, and $Y=2[\gamma_{j_2},\gamma_{j_{q+1}}]$ by part 5 of Theorem 6.3. Thus $[[X,E_0^*(\Theta_n)(x_{j_{q+1}})],\gamma_{j_2}]+[X,[\gamma_{j_2},E_0^*(\Theta_n)(x_{j_{q+1}})]]$ is equal to the coset of

$$[[X,E_0^*(\Theta_n)(x_{j_{q+1}})]\gamma_{j_2}]+[X,2[\gamma_{j_2},\gamma_{j_{q+1}})]]$$

in $H_1(L[x_1^C])$, and thus

$$[[X, E_0^*(\Theta_n)(x_{j_{q+1}}) - 2\gamma_{j_{q+1}}]\gamma_{j_2}]$$

in $H_1(L[x_1^C])$ modulo terms of lower filtration.

(9) Furthermore,

$$[[X, E_0^*(\Theta_n)(x_{j_{q+1}}) - 2\gamma_{j_{q+1}}]\gamma_{j_2}] = -[[X, \gamma_{j_{q+1}}]\gamma_{j_2}]$$

in case $X = \Lambda_n$, and

$$[[\gamma_s, E_0^*(\Theta_n)(x_{j_{q+1}}) - 2\gamma_{j_{q+1}}]\gamma_{j_2}] = 0.$$

(10) Thus if $j_2 > j_{q+1}$, then $[[X, E_0^*(\Theta_n)(x_{j_{q+1}}) - 2\gamma_{j_{q+1}}]\gamma_{j_2}]$ is one of the following.
  (a) -$[[X, \gamma_{j_{q+1}}]\gamma_{j_2}]$, or
  (b) $[[X, \gamma_{j_{q+1}}]\gamma_{j_2}]$

(11) In addition,

$$[\Lambda_n, E_0^*(\Theta_n)(x_{j_{q+1}})] = [\Lambda_n, \gamma_{j_{q+1}}]$$

as follows from 6.3, and 6.2.

Formula 5 in Theorem 6.3 is used at this point, and is stated next for the convenience of the reader. For any fixed $2 \leq i < j \leq n$, $[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = S$ where $S = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)] + 2[\gamma_j(n), \gamma_i(n)]$. This formula will be used in the expansion of $E_0^*(\Theta_n)[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]x_{j_{q+1}}]$. Thus,

$$[B, C] = [E_0^*(\Theta_n)(x_{j_q}), E_0^*(\Theta_n)(x_{j_{q+1}})] = [\gamma_{j_q}, \Lambda_n] + [\Lambda_n, \gamma_{j_{q+1}}] + 2[\gamma_{j_q}, \gamma_{j_{q+1}}].$$

Formula 5 in Theorem 6.3 is used at this point, and is stated next for the convenience of the reader. For fixed i, and j with $2 \leq i < j \leq n$,

$$[\gamma_j(n), E_0^*(\Theta_n)(x_i)] = U + V$$

where $U = [\gamma_j(n), \Lambda_n] + [\Lambda_n, \gamma_i(n)]$, and $V = 2[\gamma_j(n), \gamma_i(n)]$.

The previous formula will be used in the expansion of the element specified by an expansion of $E_0^*(\Theta_n)([X_q, x_{j_{q+1}}])$ for $X_q = [\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$. Thus, $[B, C] = [E_0^*(\Theta_n)(x_{j_q}), E_0^*(\Theta_n)(x_{j_{q+1}})] = U\text{"} + V\text{"}$ where $U\text{"} = [\gamma_{j_q}, \Lambda_n] + [\Lambda_n, \gamma_{j_{q+1}}]$, and $V\text{"} = 2[\gamma_{j_q}, \gamma_{j_{q+1}}]$.

Next, notice that

$$[A, [\gamma_q, \Lambda_n]],$$

and

$$[A, [\Lambda_n, \gamma_{q+1}]]$$

both project to 0 in $Gr_q(H_1(L[\Lambda_n^C])$. Hence the class of $[A, [B, C]]$ equal to the class of $[A, 2[\gamma_{j_q}, \gamma_{j_{q+1}}]]$.

□

Theorem 7.8. *The map*

$$E_0^*(\Theta_n) : L[x_1^B] \to L[\Lambda_n^C]$$

induces an isomorphism of Lie algebras, and thus a monomorphism. Hence Θ_n is a monomorphism.

Proof. Notice that the map $E_0^*(\Theta_n)$ sends a generator $[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$ to

$$(\pm 1)[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}] + \Omega$$

where Ω projects to an element of lower filtration degree in $H_1L[\Lambda_n^C]$ by Theorem 7.7. Hence the morphism of Lie algebras $E_0^*(\Theta_n)$ induces an isomorphism on the module of indecomposables, and thus an isomorphism of Lie algebras. The Theorem follows. □

8. On Vassiliev invariants, the mod-p descending central series, and the Bousfield-Kan spectral sequence

Let $\Gamma^n(G)$, respectively $\Gamma^{n,p}(G)$, denote the n-th stage of the descending central series for a discrete group G, respectively, the mod-p descending central series for G. Thus

(1) $\Gamma^n(G)$ is the subgroup of G generated by commutators $[\cdots[g_1, g_2], g_3], \cdots], g_t]$ for $t \geq n$ with decreasing filtration

$$G = \Gamma^1(G) \supseteq \Gamma^2(G) \supseteq \cdots ,$$

and

(2) $\Gamma^{n,p}(G)$ is the subgroup of G generated by commutators

$$[\cdots[g_1, g_2], g_3], \cdots], g_s]^{p^j}$$

for $s \cdot p^j \geq n$ with decreasing filtration

$$G = \Gamma^{1,p}(G) \supseteq \Gamma^{2,p}(G) \supseteq \cdots .$$

Let $E_0^n(G) = \Gamma^n(G)/\Gamma^{n+1}(G)$, and $E_0^{n,p}(G) = \Gamma^{n,p}(G)/\Gamma^{n+1,p}(G)$. The commutator map of sets

$$[-,-] : G \times G \to G$$

induces a natural pairing endowing

$$E_0^*(G) = \oplus_{n\geq 1} E_0^n(G)$$

with the structure of a Lie algebra while the analogous pairing for

$$E_0^{*,p}(G) = \oplus_{n\geq 1} E_0^{n,p}(G)$$

with the p-th power map $\xi : G \to G$ which induces a function

$$\xi : E_0^{n,p}(G) \to E_0^{pn,p}(G)$$

gives the structure of a restricted Lie algebra [18].

Kohno [19, 20], and Falk-Randell [15] analyzed the structure of $E_0^*(P_{n+1})$. A variation is required here. This technical variation is obtained by replacing the descending central series by the mod-p descending central series thus giving the analogous structure for $E_0^{*,p}(P_{n+1})$.

Recall that the unstable Adams spectral sequence is that obtained by filtering a simplicial group by the mod-p descending central series [11]. This variation is recorded here as it corresponds to the relationship between Vassiliev invariants, the Bousfield-Kan spectral sequence, or unstable Adams spectral sequence.

First, consider filtering via the standard descending central series. The resulting spectral sequence is labelled the Bousfield-Kan spectral sequence below. On the level of E_0^*, the morphism of simplicial groups $\Theta : F[S^1] \to \mathrm{AP}_*$ induces a map

$$E_0^*(\Theta_n) : E_0^*(F_n) \to E_0^*(P_{n+1}),$$

the subject of sections 7, and 6 on embeddings of Lie algebras here.

On the other hand, there is a second filtration, as well as a second spectral sequence obtained from the associated graded Lie algebra for the mod-p descending central series $E_0^{*,p}(F_n)$[11]. The resulting spectral sequence, the unstable Adams spectral sequence abuts to the homotopy groups of $F[S^1]$ modulo torsion prime to p.

Question: Is there a further relationship between the Vassiliev invariants of pure braids, and the homotopy groups of the 2-sphere ? Is there an informative interplay between these invariants, and homotopy theory ?

The point of the next result is to derive the structure of the Lie algebra obtained from the mod-p descending central series for the pure braid groups. Algebraic preliminaries are given next arising from work of [19, 15, 30], and the modification below for the mod-p descending central series.

Theorem 8.1. *Let*

$$1 \to A \xrightarrow{j} B \xrightarrow{p} C \to 1$$

be a short exact sequence of groups such that

- *there is a section σ for $p : B \to C$ giving $p \circ \sigma = 1_C$, and*
- *the natural action of C on $H_1(A)$ is trivial: Given b in B and a in A, then*

$$bab^{-1} = ax$$

for some x in the commutator subgroup $[A, A] = \Gamma^2(A)$.

Then there is

(1) *a split short exact sequences of Lie algebras*

$$0 \to E_0^*(A) \to E_0^*(B) \to E_0^*(C) \to 0,$$

and

(2) *a split short exact sequences of restricted Lie algebras*

$$0 \to E_0^{*,p}(A) \to E_0^{*,p}(B) \to E_0^{*,p}(C) \to 0.$$

The structure of the Lie algebra for the mod-p descending central series of the pure braid group, as well as certain other groups follows from the proof of the Proposition 8.2. A proof is analogous to the ones in [19, 15, 30]; modifications in the case of the mod-p descending central series are direct, and are given below for convenience.

Observe that $p((b \cdot \sigma(p(b^{-1}))) = 1$, and so there exists an unique element a in A with $j(a) = b \cdot \sigma p(b^{-1})$. Thus, there is a well-defined function (not necessarily a homomorphism)

$$\tau : B \to A$$

defined by the formula

$$\tau(b) = j^{-1}(b \cdot \sigma(p(b^{-1}))).$$

A useful variation of 8.1 for the mod-p descending central series is recorded next.

Proposition 8.2. *Let*

$$1 \to A \xrightarrow{j} B \xrightarrow{p} C \to 1$$

be a short exact sequence of groups such that

- *there is a section σ for $p : B \to C$ giving $p \circ \sigma = 1_C$, and*
- *the natural action of C on $H_1(A)$ is trivial: Given b in B and a in A, then*

$$bab^{-1} = ax$$

for some x in the commutator subgroup $[A, A] = \Gamma^2(A)$. (Note that x is always in $[B, B]$, but x is not necessarily an element in $[A, A]$.)

Then the following hold.

(1) *For fixed b in B, a in A, there is an element y in $[A, A]$ such that $b^{-1}ab = ay$.*
(2) *The group $[B, A]$ is a subgroup of $[A, A]$.*
(3) *The group*

$$[\Gamma^m(B), \Gamma^n(A)]$$

is a subgroup of $\Gamma^{n+m}(A)$, and the group $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$ is a subgroup of $\Gamma^{n+m,p}(A)$.

(4) *If $n \geq 1$, τ is a filtration preserving function (not necessarily a group homomorphism). That is, $\tau(\Gamma^n(B))$ is contained in $\Gamma^n(A)$, and $\tau(\Gamma^{n,p}(B))$ is contained in $\Gamma^{n,p}(A)$.*

(5) *If* $p(b) = 1$*, then* $j\tau(b) = b$*. Furthermore, there are split short exact sequences of groups*

$$1 \to \Gamma^n(A) \to \Gamma^n(B) \to \Gamma^n(C) \to 1,$$

and

$$1 \to \Gamma^{n,p}(A) \to \Gamma^{n,p}(B) \to \Gamma^{n,p}(C) \to 1.$$

(6) *If* $n \geq 1$*, there are well-defined induced functions* $\overline{\tau} : E_0^n(B) \to E_0^n(A)$*, and* $\overline{\tau} : E_0^{n,p}(B) \to E_0^{n,p}(A)$ *defined on an equivalence class of* b*,* $[b]$*, by the formula* $\overline{\tau}([b]) = [\tau(b)]$*.*

(7) *If* $n \geq 1$*, and* $[b]$ *is in the kernel of the induced homomorphism* $E_0^n(p) : E_0^n(B) \to E_0^n(C)$*, respectively* $[b]$ *is in the kernel of the induced homomorphism* $E_0^{n,p}(p) : E_0^{n,p}(B) \to E_0^{n,p}(C)$*, then* $E_0^n(j)(\overline{\tau}[b]) = [b]$*, respectively* $E_0^{n,p}(j)(\overline{\tau}[b]) = [b]$ *.*

Furthermore, there are split short exact sequences of abelian groups

$$0 \to E_0^n(A) \to E_0^n(B) \to E_0^n(C) \to 0,$$

and

$$0 \to E_0^{n,p}(A) \to E_0^{n,p}(B) \to E_0^{n,p}(C) \to 0.$$

(8) *The morphisms* $0 \to E_0^{*,p}(A) \to E_0^{*,p}(B) \to E_0^{*,p}(C) \to 0$ *are of restricted Lie algebras.*

***Remark* 8.3.** *The hypotheses above that* $bab^{-1} = ax$ *for some* x *in the commutator subgroup* $[A, A]$ *is not necessarily satisfied without the hypotheses of trivial local coefficients. Notice that if* A *is a normal subgroup of* B*, then* $a^{-1}b^{-1}ab$ *is always in* A*, but may not necessarily be in* $[A, A]$*. An example is given by the group extension* $1 \to Z/3Z \to \Sigma_3 \to Z/2Z \to 1$ *for which the commutator subgroup* $A = Z/3Z$ *is trivial, but the group* $[B, A]$ *is non-trivial for* $B = \Sigma_3$*. The hypothesis of trivial local coefficients is important in this step.*

Theorem 8.1 is a restatement of Proposition 8.2. The proof of 8.2 is based on the Hall-Witt identities together with another lemma of P. Hall both recorded in the next statement proven in [23], page 290, and [12], page 2. An additional useful statement is also given by 8.4. Recall that $[a, b] = a^{-1} \cdot b^{-1} \cdot a \cdot b$ denotes the commutator of elements a, and b in a group G, and $c^a = c^{-1} \cdot a \cdot c$.

Theorem 8.4. *For any elements of a group* G*,*

(1) $[a, b] \cdot [b, a] = 1,$
(2) $[a, b \cdot c] = [a, c] \cdot [a, b] \cdot [[a, b], c],$
(3) $[a \cdot b, c] = [a, c] \cdot [[a, c], b] \cdot [b, c],$
(4) $[[a, b], c^a] \cdot [[c, a], b^c] \cdot [[b, c], a^b] = 1,$
(5) $[[a, b], c] \cdot [[b, c], a] \cdot [[c, a], b] = [b, a] \cdot [c, a] \cdot [c, b]^a \cdot [a, b] \cdot [a, c]^b \cdot [b, c]^a \cdot [a, c] \cdot [c, a]^b,$ *and*
(6) $[a, b^n] = [a, b] \cdot [a, b]^b \cdots [a, b]^{b^{n-1}}.$

If a *is an element of* $\Gamma^{m,p}(G)$*, and* b *is an element of* $\Gamma^{n,p}(G)$*, then*

$$[a, b^p] = [a, b]^p \cdot z$$

for z *in* $\Gamma^{(m+pn),p}(G)$*.*

If A*,* B*, and* C *are normal subgroups of a group* G*, then* $[[A, B], C]$ *is contained in the subgroup generated by* $[[B, C], A]$ *and* $[[C, A], B]$*.*

The proof of Proposition 8.2 is given next.

Proof. The first statement of the proposition is one of the stated assumptions concerning the extension $1 \to A \to B \to C \to 1$. That is $b^{-1}ab = ax$ for b in B, a in A, and some x in $[A, A]$ by the assumption that the local coefficient system on $H_1(A)$ is trivial.

By part (1), $b^{-1}ab = ax$ for b in B, a in A, and some x in $[A, A]$. Thus $a^{-1}b^{-1}ab = x$, so $[B, A]$ is a subgroup of $[A, A]$. Statement (2) follows.

The proof that $[\Gamma^m(B), \Gamma^n(A)]$ is a subgroup of $\Gamma^{n+m}(A)$ is given in [15, 30]. Modifications in the case of $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$ are stated next. Since $\Gamma^1(G) = \Gamma^{1,p}(G)$ for any group G, $[\Gamma^{1,p}(B), \Gamma^{1,p}(A)] = [\Gamma^1(B), \Gamma^1(A)]$ is a subgroup of $\Gamma^2(A)$, and thus $\Gamma^{2,p}(A)$. Consider the case of $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$. Inductively assume that for $m < M$, and $n < N$ that $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$ is a subgroup of $\Gamma^{n+m,p}(A)$. That $[\Gamma^{m,p}(B), \Gamma^{n+q,p}(A)]$ is also a subgroup of $\Gamma^{n+m+q,p}(A)$ follows by induction on q via Theorem 8.4, or Appendix A of [12]: If a is an element of $\Gamma^{m,p}(G)$, and b is an element of $\Gamma^{n,p}(G)$, then $[a, b^p] = [a, b]^p \cdot z$ for z in $\Gamma^{(m+pn),p}(G)$ by Theorem 8.4. Assume that b is an element of $\Gamma^{m,p}(B)$, and that a^p is an element of $\Gamma^{n+q-1,p}(A)$, then $[b, a^p]$ is an element of $\Gamma^{m+n+q,p}(A)$.

The other cases are those in [19, 15, 30]: If $G = \Gamma^{m,p}(B)$, $K = A$, $H = \Gamma^{n+q-1,p}(A)$, then $[\Gamma^{m,p}(B), \Gamma^{n+q,p}(A)]$ is a subgroup of $\Gamma^{n+m+q,p}(A)$. A similar argument implies that if $[\Gamma^{m,p}(B), \Gamma^{n,p}(A)]$ is a subgroup of $\Gamma^{n+m,p}(A)$ for $m < M$, and $n < N$, then $[\Gamma^{m+s,p}(B), \Gamma^{n,p}(A)]$ is a subgroup of $\Gamma^{n+m+s,p}(A)$.

Recall that there is a well-defined function $\tau : B \to A$ given by the formula $\tau(b) = j^{-1}(b \cdot \sigma(p(b^{-1})))$. To prove the fourth statement, notice that the following formula holds which measures the failure of the map τ from being multiplicative:

$$\tau(xy) = (\tau(x))(\tau(y))j^{-1}([(\tau(y))^{-1}, \sigma p(x)]).$$

In addition, $[(\tau(y))^{-1}, \sigma p(x)])$ is an element of $[A, B]$ which is a subgroup of $[A, A]$. Thus for all elements x, y in B,

$$\tau(xy) = (\tau(x))(\tau(y))V,$$

as well as

$$\tau(x^p) = (\tau(x))^p W$$

where V, and W are elements of $[A, A]$ by statement (3).

Let $\lambda = \sigma \circ p$, and observe that

$$\tau(xyx^{-1}y^{-1}) = j^{-1}((xyx^{-1}y^{-1})(\lambda(y^{-1})(\lambda x^{-1})(\lambda x)(\lambda y)).$$

Furthermore, if v is an element of $\Gamma^m(B)$, and $\tau(z)$ lies in $\Gamma^n(A)$, then the commutator $[\tau(z), v]$ lies in $\Gamma^{m+n}(A)$ by part (3). Similarly, if v is an element of $\Gamma^{m,p}(B)$, and $\tau(z)$ lies in $\Gamma^{n,p}(A)$, then the commutator $[\tau(z), v]$ lies in $\Gamma^{m+n,p}(A)$ by part (3). Thus the statement that $\tau(\Gamma^n(B))$ is contained in $\Gamma^n(A)$, and $\tau(\Gamma^{n,p}(B))$ is contained in $\Gamma^{n,p}(A)$ follows by induction on n together with part (3) of the proposition. Statement (4) follows.

Statement (5) concerns $\tau : B \to A$ given by the formula $\tau(b) = j^{-1}(b \cdot \sigma(p(b^{-1})))$. By definition, if $p(b) = 1$, then $p(b^{-1}) = 1$, and $j\tau(b) = b$. Hence if if b is either in $\ker(p) \cap \Gamma^n(B)$, or $\ker(p) \cap \Gamma^{n,p}(B)$, then $\tau(b) = b$. Thus by the preceding remark as well as part (4), τ restricts to functions $\tau|_{\ker(p)\cap\Gamma^n(B)} : \ker(p) \cap \Gamma^n(B) \to \Gamma^n(A)$, and $\tau|_{\ker(p)\cap\Gamma^{n,p}(B)} : \ker(p) \cap \Gamma^{n,p}(B) \to \Gamma^{n,p}(A)$.

It follows that the homomorphisms

$$j : \Gamma^n(A) \to \ker(p) \cap \Gamma^n(B),$$

as well as

$$j : \Gamma^{n,p}(A) \to \ker(p) \cap \Gamma^{n,p}(B)$$

are group isomorphisms. Furthermore, there are exact sequences of groups $1 \to \Gamma^n(A) \to \Gamma^n(B) \to \Gamma^n(C) \to 1$, and $1 \to \Gamma^{n,p}(A) \to \Gamma^{n,p}(B) \to \Gamma^{n,p}(C) \to 1$ which are split by the existence of σ. Part (5) follows.

If $n \geq 1$, there is a well-defined induced function $\overline{\tau} : E_0^n(B) \to E_0^n(A)$ defined on an equivalence class of b by the formula $\overline{\tau}([b]) = \tau(b)$ by parts (c), and (d) together with the formula

$$\tau(xy)(\tau(x))(\tau(y))j^{-1}([(\tau(y))^{-1}, \sigma p(x)]).$$

If the class $[b]$ of an element b in $\Gamma^{n,p}(B)$ is in the kernel of the induced homomorphism $E_0^{n,p}(p) : E_0^{n,p}(B) \to E_0^{n,p}(C)$, then b is in $\Gamma^{n,p}(B)$, and $p(b)$ is in $\Gamma^{n+1,p}(C)$.

Since the natural map $\Gamma^{n+1,p}(p) : \Gamma^{n+1}(B) \to \Gamma^{n+1}(C)$ is a surjection by part (3), there is an element x in $\Gamma^{n+1,p}(B)$ such that $p(x) = p(b)$. Note that b is in $\Gamma^{n,p}(B)$, but x is in $\Gamma^{n+1,p}(B)$.

Hence x^{-1} is in $\Gamma^{n+1,p}(B)$, and bx^{-1} is a representative of the class $[b]$ in $E_0^n(B)$ with $j\tau(bx) = bx$. Hence the natural map $E_0^{n,p}(j) : E_0^{n,p}(A) \to E_0^{n,p}(B)$ is a surjection to the kernel of $\overline{\tau} : E_0^{n,p}(B) \to E_0^{n,p}(C)$, and there is a split short exact sequence $0 \to E_0^{n,p}(A) \to E_0^{n,p}(B) \to E_0^{n,p}(C) \to 0$. Statement (6) as well as the proposition follows from the above. □

The next theorem follows at once as it was checked in [6], page 251, that the local coefficient system for the extension

$$1 \to F_n \to P_{n+1} \to P_n \to 1$$

is trivial over the integers.

Theorem 8.5. *The restricted Lie algebra obtained from the mod-p descending central series for P_k is given by $\mathcal{L}_{k,p}$ the free restricted Lie algebra over $\mathbb{Z}/p\mathbb{Z}$ generated by elements $B_{i,j}$ with $1 \leq i < j \leq k$, modulo the infinitesimal braid relations:*

- **(i):** $[B_{i,j}, B_{s,t}] = 0$ *if* $\{i,j\} \cap \{s,t\} = \phi$,
- **(ii):** $[B_{i,j}, B_{i,t} + B_{t,j}] = 0$ *if* $1 \leq i < t < j \leq k$, *and*
- **(iii):** $[B_{t,j}, B_{i,j} + B_{i,t}] = 0$ *if* $1 \leq i < t < j \leq k$.

Furthermore there is a split short exact sequence of restricted Lie algebras

$$0 \to E_0^{*,p}(F_n) \xrightarrow{E_0^{*,p}(i)} E_0^{*,p}(P_{n+1}) \xrightarrow{E_0^{*,p}(d_n)} E_0^{*,p}(P_n) \to 0$$

where $E_0^{*,p}(F_n)$ is the free restricted Lie algebra generated by $B_{i,n+1}$ for $1 \leq i < n+1$. In addition, $E_0^{*,p}(P_{n+1})$ is additively isomorphic to $E_0^{*,p}(P_n) \oplus E_0^{*,p}(F_n)$.

The following is an "integrality" statement concerning embeddings of Lie algebras as do not necessarily induce an embedding after mod-p reduction.

Theorem 8.6. *If $n \geq 1$, the induced maps*

$$E_0^*(\Theta_n) : E_0^*(F_n) \to E_0^*(P_{n+1}),$$

and

$$E_0^{*,p}(\Theta_n) : E_0^{*,p}(F_n) \to E_0^{*,p}(P_{n+1})$$

are monomorphisms.

Proof. Notice that the map $E_0^*(\Theta_n)$ sends a generator $[\cdots[x_1, x_{j_1}]x_{j_2}]\cdots]x_{j_{q-1}}]x_{j_q}]$ to

$$(\pm 1)[\cdots[\Lambda_n, \gamma_{j_q}]\gamma_{j_{q-1}}]\cdots]\gamma_{j_2}]\gamma_{j_1}] + \Omega$$

where Ω projects to an element of lower filtration degree in $H_1L[\Lambda_n^C]$ by Theorem 7.7.

By 7.8, the map

$$E_0^*(\Theta_n) : L[x_1^B] \to L[\Lambda_n^C]$$

induces an isomorphism of Lie algebras over the integers. Hence, the induced map on the level of universal enveloping algebras is an isomorphism over the integers. Thus, there is an induced isomorphism on the level of universal enveloping algebras after reduction modulo p, and an isomorphism on the level of restricted Lie algebras. This suffices, and the Theorem follows. $\square$

9. On braid groups, and axioms for connected CW-complexes

The purpose of this section is to give axioms which characterize CW-complexes in terms of braid groups when viewed within the context of simplicial groups. First consider the category of groups $\mathcal{G}$, and the category of reduced simplicial groups $\mathcal{SG}$. Let $\mathcal{C}$ denote a small category.

The definition of a simplicial subgroup is used next. There are two natural choices for the definition. That these choices are equivalent is recorded below.

Consider the following axioms: Let $\mathcal{B}$ denote the smallest subcategory of $\mathcal{SG}$ which satisfies the following properties:

(1) The simplicial group AP_* is in $\mathcal{B}$.
(2) If Π, and Γ are in $\mathcal{B}$, then the coproduct $\Pi \vee \Gamma$ is in $\mathcal{B}$.
(3) If Π is in $\mathcal{B}$, and Γ is a simplicial subgroup of Π, then Γ is in $\mathcal{B}$.
(4) If Π is in $\mathcal{B}$, and Γ is a simplicial quotient of Π, then Γ is in $\mathcal{B}$.

The next result is stated in the Introduction as Theorem 1.5.

Theorem 9.1. *Let X denote simply-connected CW-complex. Then there exist an object Γ_X in $\mathcal{B}$ such that the loop space $\Omega(X)$ is homotopy equivalent to the geometric realization of Γ_X.*

First, the definition of a simplicial subgroup is required. The authors are unaware of a good reference, so additional features are stated below.

Definition 9.2. (1) A map $f : G \to H$ in $\mathcal{C}$ is a monomorphism provided whenever there are two maps $\alpha, \beta : \pi \to G$ in $\mathcal{C}$ such that

$$f \circ \alpha = f \circ \beta,$$

then

$$\alpha = \beta.$$

(2) A map $f : G \to H$ in $\mathcal{C}$ is an injection provided f is one-to-one on the underlying sets.

Next recall the following standard fact.

Proposition 9.3. *A map $f : G \to H$ in $\mathcal{G}$ is an injection if and only if f is a monomorphism.*

Proof. Assume that f is a monomorphism. It will be checked that $\ker(f)$ is trivial. Assume that there is some non-identity element x in $\ker(f)$. Define

(1) $\alpha : \mathbb{Z} \to G$ by $\alpha(n) = 1$, and
(2) $\beta : \mathbb{Z} \to G$ by $\beta(n) = x^n$.

Then $f \circ \alpha = f \circ \beta$, but $\alpha(x) \neq \beta(x)$, and so $\alpha \neq \beta$. That is a contradiction, and thus $\ker(f)$ is trivial, and so f is an injection.

Next, assume that f is an injection, and that $f \circ \alpha = f \circ \beta$. Thus $f(\alpha(x)) = f(\beta(x))$. Hence $\alpha(x) = \beta(x)$ and the proposition follows. $\square$

The next step is to check an analogous statement for reduced simplicial groups. First, two definitions should be given.

Definition 9.4. Let Γ, and Π denote reduced simplicial groups. Then Γ is a simplicial subgroup of Π provided there is a monomorphism $\phi : \Gamma \to \Pi$ in $\mathcal{SG}$.

Definition 9.5. Let Γ, and Π denote reduced simplicial groups. Then the pair (Π, Γ) is a simplicial group pair provided there is a morphism in $\mathcal{SG}$ given by $\phi : \Gamma \to \Pi$ which is a degree-wise injection of groups.

Proposition 9.6. *Given simplicial groups Γ, and Π, the following are equivalent.*

1. Γ *is a simplicial subgroup of* Π.
2. *The pair* (Π, Γ) *is a simplicial group pair.*

Proof. Assume that $f : \Gamma \to \Pi$ is a monomorphism in $\mathcal{SG}$. It will be checked that in each simplicial degree n, $\ker(f)$ is trivial. Assume that n is the minimal degree for which there is a non-trivial element x in $\ker(f)$.

Let

$$\bar{G}\langle x\rangle$$

denote the simplicial closure of x as given in [8]. Thus $\bar{G}\langle x\rangle$ is both a simplicial group, and the natural morphism of simplicial groups $\beta : \bar{G}\langle x\rangle \to \Gamma$ satisfies

$$\beta(x) = x.$$

Thus, there are morphisms in $\mathcal{SG}$ given by

1. $\alpha : \bar{G}\langle x\rangle \to \Gamma$ by $\alpha(x) = 1$, and
2. $\beta : \bar{G}\langle x\rangle \to \Gamma$ by $\beta(x) = x$.

Then $f \circ \alpha = f \circ \beta$, but $\alpha(x) \neq \beta(x)$, and so $\alpha \neq \beta$.

This statement contradicts the fact that f is monomorphism. Thus $\ker(f)$ is trivial, and f is an injection in each degree.

Next, assume that f is an injection in each degree, and that $f \circ \alpha = f \circ \beta$. Thus in each degree, it follows that $f(\alpha(x)) = f(\beta(x))$. Hence $\alpha(x) = \beta(x)$ and the proposition follows. □

10. Proof of Theorem 1.5

The simplicial group AP_* is in $\mathcal{B}$ by axiom 1. By Theorem 1.2, there is a morphism of simplicial groups

$$\Theta : F[S^1] \to \mathrm{AP}_*$$

which is a degree-wise injection. Thus $F[S^1]$ is a simplicial subgroup of AP_* by 9.6. Hence $F[S^1]$ is in $\mathcal{B}$ by axiom 3.

Notice that coproducts, $\mathrm{AP}_* \vee \mathrm{AP}_*$, as well as $F[S^1] \vee F[S^1]$ are in $\mathcal{B}$ by axiom 2. Since $F[S^n]$ is a subgroup of $F[S^1] \vee F[S^1]$ for $n \geq 1$, $F[S^n]$ is in $\mathcal{B}$. By passage to coproducts, $\vee_{n\epsilon T} F[S^n]$ is in $\mathcal{B}$ for any set T.

The next statement concerning push-outs is the simplicial analogue of a classical result of J. H. C. Whitehead [16, 22]. found in [29].

Proposition 10.1. *Let G_0, G_1, and G_2 be simplicial groups in $\mathcal{B}$ together with morphisms $\alpha : G_0 \to G_1$, and $\beta : G_0 \to G_2$ in $\mathcal{B}$. Then*

1. *the push-out* Π *of* $\alpha : G_0 \to G_1$, *and* $\beta : G_0 \to G_2$ *is in* $\mathcal{B}$, *and*
2. *if both α, and β are monomorphisms in $\mathcal{B}$, the classifying space of Π is the push-out of the classifing space construction of $\alpha : G_0 \to G_1$, and $\beta : G_0 \to G_2$.*

Let Y be the cofibre of a map $a : \vee_{n\epsilon T} S^n \to X$, and assume that ΩX is homotopy equivalent to a simplicial group G in $\mathcal{B}$. Then consider the push-out of groups

$$\begin{array}{ccc} F[\vee_{n\epsilon T} S^{n-1}] & \xrightarrow{f} & G \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \Theta} \\ \vee_S \mathrm{AP}_* & \longrightarrow & \Gamma \end{array}$$

where f is the natural extension of the attaching map a, and g can be chosen to be a monomorphism. If f is a monomorphism, the push-out Γ lies in the category $\mathcal{B}$. Since $\vee_S \mathrm{AP}_*$ is contractible, the geometric realization of G is homotopy equivalent to ΩY.

It will be checked next that f may be assumed to be a monomorphism. First assume that G' is homotopy equivalent to ΩX and

$$f' : F[\vee_{n\epsilon T} S^{n-1}] \to G'$$

represents the looping of the attaching map. Observe that $F[\vee_{n\epsilon T} S^{n-1}]$ embeds in $\vee_S \mathrm{AP}_*$ via the injection

$$g : F[\vee_{n\epsilon T} S^{n-1}] \to \vee_S \mathrm{AP}_*.$$

Next, recall that $\mathcal{B}$ contains products as it contains both coproducts, and quotients. Let

$$G = G' \times (\vee_S \mathrm{AP}_*),$$

and let f be the injection

$$f' \times g : F[\vee_{n\epsilon T} S^{n-1}] \to G.$$

Notice that G is homotopy equivalent to ΩX because AP_* is contractible, and that G is in $\mathcal{B}$. Hence f may be replaced by $f' \times g$, and thus assumed to be a monomorphism.

The result follows by 10.1.

11. Appendix: a sample computation

The purpose of this section is to list a sample computation for the values of Θ_3.

(1) $\Theta_3([x_1, x_2]x_2]x_3]) = [\Lambda_3, \gamma_3]\gamma_2]\gamma_2]$,
(2) $\Theta_3([x_1, x_2]x_3]x_2]) \cong -[\Lambda_3, \gamma_2]\gamma_3]\gamma_2] + 2[\Lambda_3, \gamma_3]\gamma_2]\gamma_2]$ modulo decomposable elements, and
(3) $\Theta_3([x_1, x_3]x_2]x_2]) \cong [\Lambda_3, \gamma_2]\gamma_2]\gamma_3] - 2[\Lambda_3, \gamma_2]\gamma_3]\gamma_2] + 2[\Lambda_3, \gamma_3]\gamma_2]\gamma_2]$ modulo decomposable elements.

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