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Topics:

1) How to get homotopy from braids?
2) Simplicial groups and \Delta-groups.
3) A combinatorial model for \Omega S^2.
4) Results.
5) Remarks.

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1. How to get homotopy from braids?

- Consider the sequence of braids \( \{B_{n+1}\}_{n \geq 0} \). Let a braid \( \beta \) be labelled by 0th-strand, 1st-strand, \ldots, \( n \)th-strand.

- \( d_i \beta \) given by deleting \( i \)th-strand. (face operations)

- \( s_i \beta \) given by doubling \( i \)th-strand. (degeneracy operations)

\[ \implies \text{simplicial set} \ \{B_{n+1}\}_{n \geq 0} \ \text{and simplicial group} \ \{P_{n+1}\}_{n \geq 0}, \]

where \( P_n \) is the pure braid group.

- In short, simplicial structure is from deleting-doubling system.
• The **deleting rule** applies to \( \{B_{n+1}(M)\}_{n \geq 0} \), the braids over any manifold \( M \). This \( \implies \) \( \Delta \)-structure on \( \{B_{n+1}(M)\}_{n \geq 0} \) for any \( M \), namely **only having faces**.

• Existence of doubling \( \iff \) \( M \) has a nowhere zero vector field: Given a manifold \( M \), let

\[
F(M, n + 1) = \{(z_0, \ldots, z_n) \mid z_i \neq z_j \text{ for } i \neq j\}.
\]

Recall that \( B_{n+1}(M) = \pi_1(F(M, n + 1)/\Sigma_{n+1}) \), considered as a subset of the fundamental groupoid of \( F(M, n + 1) \) consisting of (homotopy classes) of paths starting from the base-point and ending with a coordinate permutation of the base-point.

**Doubling** is induced from maps \( s_i : F(M, n + 1) \to F(M, n + 2) \)

\[
s_i(z_0, \ldots, z_n) = (z_0, \ldots, z_{i-1}, z_i, z_i', z_{i+1}, \ldots, z_n),
\]

where \( z_i' \) is close to \( z_i \) but not equal to any of \( z_j \), continuously dependent on \( z_j \).

• For instance, \( \{P_{n+1}(S^2)\}_{n \geq 0} \) is only a \( \Delta \)-group, while the sequence of Artin pure braids \( \{P_{n+1}(D^2)\}_{n \geq 0} \) is a simplicial group.
2. Simplicial Groups and $\Delta$-Groups

A simplicial group $G$ means a sequence of groups $G = \{G_n\}_{n \geq 0}$ with face homomorphisms $d_i : G_n \rightarrow G_{n-1}$ and degeneracy homomorphisms $s_i : G_n \rightarrow G_{n+1}$, $0 \leq i \leq n$, such that the simplicial identities: $d_i d_j = d_j d_{i+1}$ for $i \geq j$, $s_i s_j = s_{j+1} s_i$ for $i \leq j$ and

$$d_j s_i = \begin{cases} 
  s_{i-1} d_j & j < i \\
  \text{id} & j = i, i + 1 \\
  s_i d_{j-1} & j > i + 1 
\end{cases}$$

hold.

A $\Delta$-group $G$ means a sequence of groups $G = \{G_n\}_{n \geq 0}$ with face homomorphisms $d_i : G_n \rightarrow G_{n-1}$, $0 \leq i \leq n$, such that the simplicial identity $d_i d_j = d_j d_{i+1}$ for $i \geq j$ holds.

- If $G$ is a $\Delta$-group such that each $G_n$ is abelian, then, under

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i,$$

$(G, \partial)$ is a chain complex. Thus the faces $d_i$ can be considered as partial differentials.
Let $G$ be a $\Delta$-group. Define

$$N_n G = \bigcap_{i>0} \ker(d_i : G_n \to G_{n-1})$$

called Moore chains

This gives a chain complex of (non-commutative in general) groups:

$$\cdots \to N_{n+1} G \xrightarrow{d_0} N_n G \xrightarrow{d_0} N_{n-1} G \to \cdots ,$$

that is, $d_0 \circ d_0$ is the trivial homomorphism in $NG = \{N_n G\}_{n \geq 0}$.

(Follows from $d_0 d_0 = d_0 d_1$.)

- Moore cycles: $Z_n G = \bigcap_{i=0}^n \ker(d_i : G_n \to G_{n-1})$
- Moore boundaries: $B_n G = d_0(N_{n+1} G)$
- Moore homotopy groups:

$$\pi_n(G) = Z_n G / B_n G = H_n(NG, d_0)$$
• If $G$ is a simplicial group, then, by the classical Moore Theorem, $\pi_n(G)$ (defined as above) is the same as $\pi_n(|G|)$, the $n$-th homotopy group of its geometric realization $|G|$.

• Let $X$ be any path-connected loop space. Then there is a simplicial group $G$ such that $|G|$ is (weak) homotopy equivalent to $\Omega X$. Thus simplicial groups are combinatorial models of loop spaces.

• It is possible that there are two simplicial groups $G$ and $G'$ such that $|G| \simeq |G'|$ but, in each dimension, $G_n$ and $G'_n$ have different group-theoretic properties such as $G_n$ could be a free group while $G'_n$ could be a perfect group. Thus a good choice of simplicial group model may result in some special properties.
3. A Combinatorial Model for $\Omega S^2$

Pursue Birman’s orange book:

$$\hat{F}_n = F(x_0, x_1, \ldots, x_n)/\langle x_0x_1\cdots x_n \rangle,$$

the quotient of $F_{n+1}$ by the single relation $x_0x_1\cdots x_n = 1$.

- $\hat{F}_n$ is a free group of rank $n$. In geometry, $\hat{F}_n$ is the fundamental group of $S^2$ punctured by $(n+1)$-points.

Add canonical simplicial structure to $\{\hat{F}_n\}_{n \geq 0}$ by:

$$d_i: \hat{F}_n \rightarrow \hat{F}_{n-1} \quad x_i \mapsto 1, \quad x_j \mapsto x_j \ (j < i), \quad x_j \mapsto x_{j-1} \ (j > i).$$

$$s_i: \hat{F}_n \rightarrow \hat{F}_{n+1} \quad x_i \mapsto x_ix_{i+1}, \quad x_j \mapsto x_j(j < i), \quad x_j \mapsto x_{j+1}(j > i).$$

In geometry, the $i$-face $d_i$ is obtained by forgetting $i$ th hole and the $i$-degeneracy is obtained by doubling $i$ th hole.

**Theorem.** $\hat{F} = \{\hat{F}_n\}_{n \geq 0}$ is a simplicial group. Moreover the geometric realization $|\hat{F}| \simeq \Omega S^2$. 
Results

• Centers of the combinatorially given groups.

Let $G(n)$ be the quotient group of $\hat{F}_n$ subject to the iterated commutator relations:

\[(1) \quad [x_{i_1}^{\epsilon_1}, x_{i_2}^{\epsilon_2}, \ldots, x_{i_t}^{\epsilon_t}] = 1,\]

where $\epsilon_j = \pm 1$, every $x_j$ occurs at least once in the bracket and the bracket runs over all possible commutator brackets.

**Theorem.** $\pi_n(S^2)$ is isomorphic to the center of the group $G(n)$.

Fixed-sets of the pure braid group actions

The reduced Artin representation of $B_n$ on $\hat{F}_n$ induces an $B_n$-action on the quotient group $G(n)$.

Theorem. $\pi_n(S^2) = Z(G(n))$ is the fixed set of the pure braid group $P_n$ action on $G(n)$.

•Brunnian braids

Let Brun$_n$(M) denote the group of n-strand Brunnian braids over the manifold M.

**Theorem.** There is an exact sequence of groups

\[ 1 \longrightarrow \text{Brun}_n(S^2) \longrightarrow \text{Brun}_n(D^2) \]

\[ f_* : \text{Brun}_n(S^2) \longrightarrow \pi_{n-1}(S^2) \longrightarrow 1 \]

for \( n \geq 5 \), where \( f_* \) is induced from the canonical embedding \( f : D^2 \to S^2 \).

Thus the torsion homotopy groups of \( S^2 \) (or \( S^3 \)) are the invariants for measuring the difference of the Brunnian braids between \( S^2 \) and \( D^2 \).
• **Homology of Brunnian braids**

There is a differential on the sequence of the classical Brunnian braids \( \{ \text{Brun}_n(D^2) \} \), which is essentially induced from complex-conjugation operation on configuration spaces together with mirror reflection of braids. This makes \( \{ \text{Brun}_n(D^2) \} \) is a chain complex of non-commutative groups.

**Theorem** For all \( n \) there is an isomorphism of groups

\[
H_n(\text{Brun}(D^2)) \cong \pi_n(S^2).
\]

Remarks

• Problem 23 in Birman’s orange book is equivalent to find a basis of the Moore cycles (for $\Omega S^2$). Since there is description for the Moore boundary already, a solution to Birman’s problem will give a combinatorial determination of (general) homotopy groups of the sphere.

• $\text{Brun}_n(S^2)/\text{Brun}_n(D^2)$ is finite abelian for $n \geq 5$. **Problem:** Determine the index of $\text{Brun}_n(D^2)$ in $\text{Brun}_n(S^2)$, that is the order of $\text{Brun}_n(S^2)/\text{Brun}_n(D^2)$.

• There are more connections between homotopy and braids such as Lie algebras associated to Vassiliev invariants in work of T. Kohno. (**Ref:** F. R. Cohen and J. Wu, *On braid groups, free groups, and the loop space of the 2-sphere*, preprint.)