Braid Groups

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Braid Groups

Definition

- The *n th ordered configuration space* $F(M, n)$ of a space M is defined by

$$F(M,n) = \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\},$$

  and has the topology of a subspace of the product space M^n.
- Σ_n acts on $F(M, n)$ by permuting coordinates, that is,

$$\begin{aligned} \sigma \cdot (x_1, \dots, x_n) &= (x_{1\cdot\sigma}, \dots, x_{n\cdot\sigma}) \\ &= (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}). \end{aligned}$$

- The orbit space $B(M, n) = F(M, n)/\Sigma_n$ is called the n *th unordered configuration space*.

Remarks

- If M is Hausdorff, then $F(M,n)$ is an open subspace of M^n.
- $F(M,n)$ is the largest subspace of M^n with free S_n-action.
- The complement $\Delta = M^n \setminus F(M,n)$ is the fat diagonal. If $M = \mathbb{R}$ or $\mathbb{C}$, this is a *hyperplane arrangement*.
- One of the most important cases is $M = \mathbb{C}$. In this case

$$\Delta = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid \text{discriminant } \prod_{i<j}(z_i - z_j)^2 = 0.\}$$

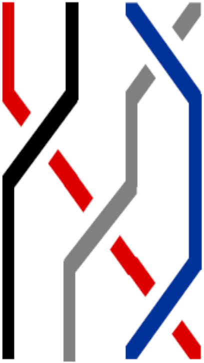
- $B(\mathbb{C}, n) = \{(a_1, \ldots, a_n) \in \mathbb{C}^n \mid f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \text{ has } n \text{ distinct roots } \}.$

Braids

Consider the covering $p\colon F(M,n) \longrightarrow B(M,n) = F(M,n)/\Sigma_n$ with fibre Σ_n. Choose a base point $(q_1, q_2, \cdots, q_n)$ for $F(M,n)$.

- Let $\omega\colon S^1 \to B(M,n)$ be a loop. Then there is a lifting path $\lambda\colon [0,1] \to F(M,n)$ such that $\lambda(0) = (q_1, q_2, \cdots, q_n)$, $\lambda(1) = (q_{\sigma(1)}, \cdots, q_{\sigma(n)})$ for some $\sigma \in \Sigma_n$ and $p(\lambda) = \omega$. Thus
$$\lambda(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t))$$
with $\lambda_i(t) \neq \lambda_j(t)$ for $i \neq j$ and $0 \leq t \leq 1$. We obtain n strings $\lambda_i(t)$ in the cylinder $M \times I$ starting at q_i and ending with $q_{\sigma(i)}$ for some σ.
- The multiplication is given by the composition of strings.
- The pure braids are n strings $\lambda_i(t)$ in $M \times I$ starting at q_i and ending with q_i.

Braids



fibre bundles

- A *bundle* means a triple (E, p, B), where $p\colon E \to B$ is a (continuous) map. B : *base space*, E : *total space*, the map p *bundle projection*. For each $b \in B$, $p^{-1}(b)$ called a *fibre*.
- A *fibre bundle* $p\colon E \to B$ is a "locally trivial" bundle with a "fixed fibre" F. More precisely, for any $x \in B$, $\exists$ an open neighborhood U of x such that $p^{-1}(U)$ is a trivial bundle, i.e., $\exists$ a homeomorphism $\phi_U\colon p^{-1}(U) \to U \times F$ such that the diagram

$$\begin{array}{ccc} U \times F & \xrightarrow{\phi_x} & p^{-1}(U) \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle p} \\ U & = & U \end{array}$$

commutes, i.e., $p(\phi(x', y)) = x'$ for any $x' \in U$ and $y \in F$.

Fadell-Neuwirth Theorem

- Let $Q_m = \{q_1, \ldots, q_m\}$ be a set of m fixed, pairwise different points in M. Let M be a (path-connected) manifold without boundary. Then the canonical projection

$$F(M,n) \longrightarrow F(M,r) \quad (x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_r)$$

is a fibre bundle with fibre $F(M \smallsetminus Q_r, n-r)$.

- Note: $F(M,n) \simeq F(M \smallsetminus \partial M, n)$.
- E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. **10** (1962), 111–118.

Ideas of Proof

- Special case $M \setminus Q_{n-1} \longrightarrow F(M,n) \xrightarrow{p} F(M,n-1)$.
- For $(a_1,\ldots,a_{n-1}) \in F(M,n-1)$, choose **small neighborhood** U_i of a_i with $U_i \cap U_j = \emptyset$ for $i \neq j$. $U = U_1 \times \cdots \times U_{n-1}$.

$$p^{-1}(U) = \{(x_1,\ldots,x_n) \in U \times M \mid x_n \neq x_i\}.$$

- $\phi\colon U \times (M \setminus \{a_1,\ldots,a_{n-1}\}) \xrightarrow{\cong} p^{-1}(U)$ $(x_1,\ldots,x_{n-1},y) \mapsto (x_1,\ldots,x_{n-1},z=\phi_{x_i}(y))$. Condition: $y \neq a_i$, $z \neq x_i$.
- Construct $\phi_{x_i}\colon U_i \xrightarrow{\cong} U_i$ such that $\phi_{x_i}|_{\partial U_i} = \mathrm{id}$ and $\phi_{x_i}(a_i) = x_i$. Extend to $\phi_{x_i}\colon M \xrightarrow{\cong} M$.

Cohomology

- **Problem:** Determine the cohomology $H^*(F(M,n))$ and $H^*(B(M,n))$.
- Fibre bundle $(F(M,n-1)\times M, F(M,n)) \longrightarrow F(M,n-1)$ with fibre $(M, M\smallsetminus Q_{n-1}) \simeq \bigvee^{n-1} S^m$, where $m=\dim M$.
- If M is oriented, then $H^*(F(M,n-1)\times M, F(M,n))$ is the free $H^*(F(M,n-1))$-module generated by $\tau_{i,n}$, $1\leq i\leq n-1$.
- $H^*(F(M\times\mathbb{R},n))$ and other cases were known.
- Good Reference: F. Cohen and L. Taylor, topology of configuration spaces, preprint.

Configuration Space with labels

- Let M be a manifold, let M_0 be a submanifold of M and let X be a pointed space with based-point $*$. The configuration space

$$C(M, M_0; X) = \coprod_{k=1}^{\infty} F(M, k) \times_{\Sigma_k} X^k / \approx,$$

where $\approx$ is generated by

$$(a_1, ..., a_k; x_1, ..., x_k) \approx (a_1, ..., a_{k-1}; x_1, ..., x_{k-1})$$

if $a_k \in M_0$ or $x_k = *$.

Model for Mapping Spaces

- Let M be an m-manifold and let W be a m-manifold without boundary which contains M, e.g. $W = M$ if M is closed, or $W = M \cup \partial M \times [0, 1)$ if M has boundary.
- Let ξ be the principal $O(m)$-bundle of the tangent bundle of W.
- Let $\xi[S^m X]$, $E\xi[S^m X] = E\xi \times_{O(m)} S^m X$, is the associated bundle, where $O(m)$ acts diagonally on $S^m X = S^m \wedge X$, trivially on X and canonically on $S^m \cong R^m \cup \{\infty\}$.
- Let $\Gamma_{\xi[S^m X]}(B, B_0)$ be the space of cross sections of $\xi[S^m X]$ which are defined on B and take values at $\infty \wedge X$ on B_0 for each subspace pair (B, B_0) in W.

Theorem

- Let M be a smooth compact manifold and let M_0 be a smooth compact submanifold of M. If M/M_0 or X is path connected, then there is a (weak) homotopy equivalence

$$C(M, M_0; X) \to \Gamma_{\xi[S^m X]}(W - M_0, W - M).$$

- Examples: $C(\mathbb{R}^n; X) \simeq \Omega^n \Sigma^n X$ if X is path-connected.
- References:
  - G. Segal, *Configuration spaces and iterated loop spaces* Invent.Math. 21 (1973), 213-221.
  - D.McDuff, *Configuration spaces of positive and negative particles* Topology 14 (1975), 91-107.

Homology

- **Problem**: Determine $H_*(C(M, M_0; X))$.
- C.-F.Bödigheimer,F.R.Cohen and L.Taylor, *On the homology of configuration spaces* topology 28 (1989), 111-123.
- J. Wu, *On the homology of configuration spaces* $C((M, M_0) \times \mathbf{R}^n; X)$, Math. Z., 22(1998), 235-248.

Braids via configuration spaces

- $B_n = \pi_1(B(\mathbb{C}, n))$, $P_n = \pi_1(F(\mathbb{C}, n))$.
- Covering space $F(\mathbb{C}, n) \to B(\mathbb{C}, n) = F(\mathbb{C}, n)/\Sigma_n$.
- There is a short exact sequence

$$P_n \hookrightarrow B_n \twoheadrightarrow \Sigma_n.$$

$K(\pi,1)$ spaces

- A space X is call a $K(\pi,1)$ space if X is path-connected, $\pi=\pi_1(X)$ and $\pi_n(X)=0$ for $n\geq 2$.
- $B(\mathbb{C},n)=K(B_n,1)$ and $F(\mathbb{C},n)=K(P_n,1)$.
- Proof of $F(\mathbb{C},n)=K(P_n,1)$ by induction: For $n=1$, $F(\mathbb{C},n)=\mathbb{C}\simeq *$. Consider the fibration

$$\mathbb{C}\smallsetminus Q_n \longrightarrow F(\mathbb{C},n+1)\longrightarrow F(\mathbb{C},n)$$

and use $\mathbb{C}\smallsetminus Q_n\simeq \bigvee^{n} S^1 = K(F_n,1)$.

Decomposition Theorem

- For $n \geq 2$, the fibre sequence $SO(2) \to F(\mathbb{C}, n) \to ESO(2) \times_{SO(2)} F(\mathbb{C}, n)$ induces

$$\begin{aligned} F(\mathbb{C}, n) &\simeq SO(2) \times \left(ESO(2) \times_{SO(2)} F(\mathbb{C}, n)\right) \\ &\simeq S^1 \times F(\mathbb{C} \smallsetminus Q_2, n-2). \end{aligned}$$

- For $n \geq 3$, the fibre sequence $SO(3) \to F(S^2, n) \to ESO(3) \times_{SO(3)} F(S^2, n)$ induces

$$\begin{aligned} F(S^2, n) &\simeq SO(3) \times \left(ESO(3) \times_{SO(3)} F(S^2, n)\right) \\ &\simeq SO(3) \times F(S^2 \smallsetminus Q_3, n-3). \end{aligned}$$

- The center of the group $\pi_1(F(\mathbb{C} \smallsetminus Q_i, n))$ is trivial for $i \geq 2$ and $n \geq 1$.

Proof of the Decomposition Theorem

(1). Consider the (homotopy) commutative diagram

$$
\begin{array}{ccccccc}
* & \longrightarrow & F(\mathbb{C}\smallsetminus Q_2, n-2) & = & F(\mathbb{C}\smallsetminus Q_2, n-2) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
SO(2) & \longrightarrow & F(\mathbb{C}, n) & \longrightarrow & ESO(2)\times_{SO(2)} F(\mathbb{C}, n) & \longrightarrow & BSO(2) \\
\| & & \downarrow & & \downarrow & & \| \\
SO(2) & \xrightarrow{g} & F(\mathbb{C}, 2) & \longrightarrow & ESO(2)\times_{SO(2)} F(\mathbb{C}, 2) & \longrightarrow & BSO(2).
\end{array}
$$

Define $\phi\colon S^1 \to F(\mathbb{C}, 2)$ by $\phi(z) = (z, -z)$. Then ϕ is an $SO(2)$-equivariant homotopy equivalence. Thus

$$ESO(2) \times_{SO(2)} F(\mathbb{C}, 2) \simeq ESO(2) \times_{SO(2)} S^1$$

is contractible. It follows that the maps

$$g\colon SO(2) \to F(\mathbb{C}, 2)$$

$$F(\mathbb{C} \smallsetminus Q_2, n-2) \to ESO(2) \times_{SO(2)} F(\mathbb{C}, n)$$

are homotopy equivalences. Hence the inclusion $SO(2) \to F(\mathbb{C}, n)$ admits a retraction and so

$$F(\mathbb{C}, n) \simeq SO(2) \times \big(ESO(2) \times_{SO(2)} F(\mathbb{C}, n)\big) \simeq S^1 \times F(\mathbb{C} \smallsetminus Q_2, n-2),$$

which is assertion (1).

(2). The proof for assertion (2) is similar to that for assertion (1) by considering the (homotopy) commutative diagram

$$
\begin{array}{ccccccc}
* & \longrightarrow & F(S^2 \smallsetminus Q_3, n-3) & = & F(S^2 \smallsetminus Q_3, n-3) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
SO(3) & \longrightarrow & F(S^2, n) & \longrightarrow & ESO(3) \times_{SO(3)} F(S^2, n) & \longrightarrow & BSO(3) \\
\| & & \downarrow & & \downarrow & & \| \\
SO(3) & \longrightarrow & F(S^2, 3) & \longrightarrow & ESO(3) \times_{SO(3)} F(S^2, 3) & \longrightarrow & BSO(3)
\end{array}
$$

and using the fact that $F(S^2, 3)$ is $SO(3)$-homotopy equivalent to $SO(3)$.

The proof is given by induction on n. When $n = 1$,

$$\pi_1(F(\mathbb{C} \smallsetminus Q_i, 1)) = \pi_1(\mathbb{C} \smallsetminus Q_i) = F_i$$

is the free group of rank i. Since $i \geq 2$, $Z(F_i) = \{1\}$. Thus the assertion holds for $n = 1$. Suppose that the assertion holds for $n = k$. Since the Fadell-Neuwirth fibration

$$\mathbb{C} \smallsetminus Q_{k+i} \longrightarrow F(\mathbb{C} \smallsetminus Q_i, k+1) \xrightarrow{d_1} F(\mathbb{C} \smallsetminus Q_i, k)$$

admits a cross-section, there is a short exact sequence of groups

$$F_{k+i} = \pi_1(\mathbb{C} \smallsetminus Q_{k+i}) \hookrightarrow \pi_1(F(\mathbb{C} \smallsetminus Q_i, k+1)) \xrightarrow{d_{1*}} \pi_1(F(\mathbb{C} \smallsetminus Q_i, k)).$$

Let $w \in Z(\pi_1(F(\mathbb{C} \setminus Q_i, k+1)))$. Then

$$d_{1*}(w) \in Z(\pi_1(F(\mathbb{C} \setminus Q_i, k)))$$

because d_{1*} is onto. By induction $Z(\pi_1(F(\mathbb{C} \setminus Q_i, k))) = \{1\}$, thus $d_{1*}(w) = 1$ and so

$$w \in \pi_1(\mathbb{C} \setminus Q_{k+i}) = F_{k+i}.$$

Since $w \in Z(\pi_1(F(\mathbb{C} \setminus Q_i, k+1)))$, one gets $w \in Z(F_{k+i}) = \{1\}$. It follows that $w = 1$. The induction is finished and hence the result.

Chow's Theorem on center of braid groups

- For $n \geq 2$, $Z(P_n(D^2)) \cong \mathbb{Z}$ with a generator represented by the $SO(2)$-orbit of the base-point in $F(\mathbb{C}, n)$. Thus $Z(P_n(D^2))$ is generated by the full-twist braid $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$.
- For $n \geq 3$, $Z(P_n(S^2)) \cong \mathbb{Z}/2$ with a generator represented by the $SO(2)$-orbit of the base-point in $F(S^2, n)$.
- The decomposition theorem gives a new proof of W. L. Chow, *On the algebraic braid group*, Ann. of Math. 49(1948), 654-658.

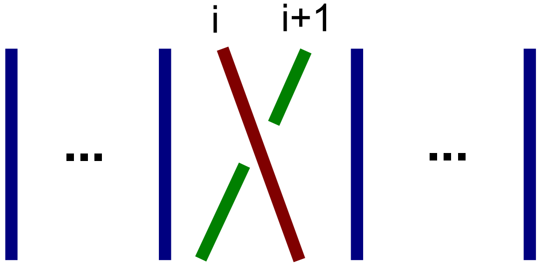
Braids via generators and relations

- B_n: generators: $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$; defining relations:
  - $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.
  - $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$.
- P_n: generators $A_{i,j}$, $1 \leq i < j \leq n$. Relations:

$$A_{r,s}^{-1} A_{i,j} A_{r,s} = \begin{cases} A_{i,j} & \text{if} \quad 1 \leq r < s < i < j \leq n, \\ A_{i,j} & \text{if} \quad 1 \leq i < r < s < j \leq n, \\ A_{r,j} A_{i,j} A_{r,j}^{-1} & \text{if} \quad 1 \leq r < s = i < j \leq n, \\ (A_{i,j} A_{s,j}) A_{i,j} (A_{i,j} A_{s,j})^{-1} & \text{if} \quad 1 \leq r = i < s < j \leq n, \\ [A_{r,j}, A_{s,j}] A_{i,j} [A_{r,j}, A_{s,j}]^{-1} & \text{if} \quad 1 \leq r < i < s < j \leq n, \end{cases}$$

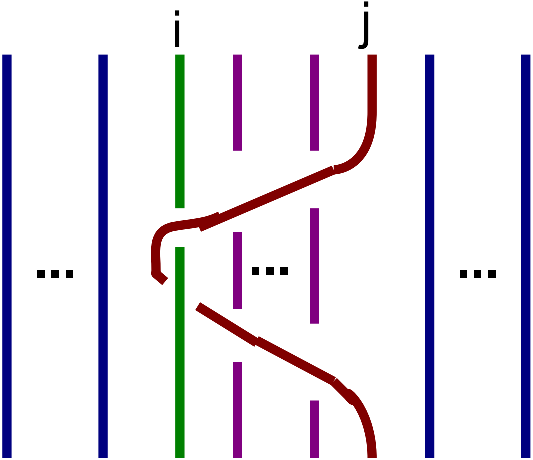
where $[a, b] = aba^{-1} b^{-1}$.

σ_i



σ_i

$A_{i,j}$



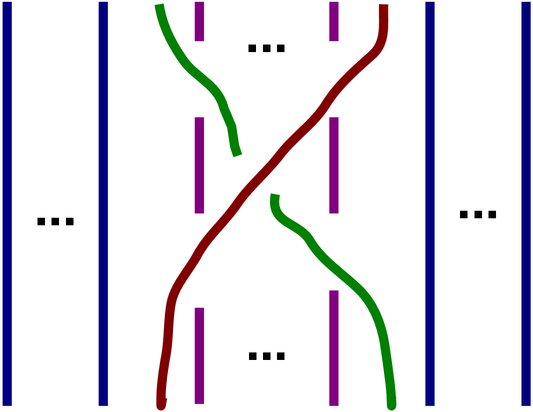
Birman-Ko-Lee Presentation for B_n

- $\sigma_{s,t} = (\sigma_{t-1} \cdots \sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1} \cdots \sigma_{t-1}^{-1})$, where $1 \leq s < t \leq n$. Set $\sigma_{s,t} = \sigma_{t,s}$.
- Generators: $\sigma_{s,t}$, $1 \leq s < t \leq n$.
- Relations:

$$\begin{aligned} \sigma_{s,t}\sigma_{q,r} &= \sigma_{q,r}\sigma_{s,t} \quad \text{if} \quad (t-r)(t-q)(s-r)(s-q) > 0, \\ \sigma_{s,t}\sigma_{r,s} &= \sigma_{r,t}\sigma_{s,t} \\ &= \sigma_{r,s}\sigma_{r,t} \quad \text{if} \quad 1 \leq r < s < t \leq n. \end{aligned}$$

- $A_{s,t} = \sigma_{s,t}^2$.
- Ref: J. S. Birman and T. E. Brendle,*braids: a survey*, 2004.

$\sigma_{s,t}$



Mapping class groups

- Let $S = S_{g,b,n}$ denote a 2-manifold of genus g with b boundary and n punctures.
- $\mathrm{Diff}^+(S) :=$ the group of all orientation preserving diffeomorphisms of S which fix the boundary points pointwise and permute the punctured points, with compact open topology. So $\mathrm{Diff}^+(S)$ is a topological group.
- **Mapping class group** $\mathcal{M} = \mathcal{M}_{g,b,n}$ of S is $\pi_0(\mathrm{Diff}^+(S))$, i.e. the quotient of $\mathrm{Diff}^+(S)$ modulo its subgroup of all diffeomorphisms of S which are isotopic to the identity relative to ∂S.
- $\mathrm{Diff}^+(S_{g,b,\hat{n}})$ be the subgroup of $\mathrm{Diff}^+(S)$ consisting of diffeomorphisms that fix the punctures pointwise. $\mathcal{M}_{g,b,\hat{n}} = \pi_0(\mathrm{Diff}^+(S_{g,b,\hat{n}})$.

Theorem

- Theorem: $B_n = \mathcal{M}_{0,1,n}$, $P_n = \mathcal{M}_{0,1,\hat{n}}$.
- Ideas of Proof: Commutative diagram of fibre sequences

$$\begin{array}{ccccc}
\mathrm{Diff}^+(S_{0,1,\hat{n}}) & \rightarrowtail & \mathrm{Diff}^+(S_{0,1,n}) & \longrightarrow & \Sigma_n \\
\| & & \downarrow & & \downarrow \\
\mathrm{Diff}^+(S_{0,1,\hat{n}}) & \rightarrowtail & \mathrm{Diff}^+(S_{0,1,0}) & \xrightarrow{E} & F(D^2, n) \\
 & & \downarrow & & \downarrow \\
 & & B(D^2, n) & = & B(D^2, n)
\end{array}$$

with $\mathrm{Diff}^+(S_{0,1,0}) \simeq *$, $E(h) = (h(q_1), \ldots, h(q_n))$.

- Details: J. S. Birman and T. E. Brendle,*braids: a survey*.

Artin Representation

- $\pi_1(D^2 \smallsetminus Q_n) = F_n$ the free group of rank n.
-

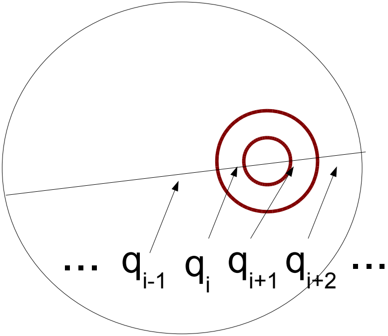
$$\begin{array}{llll} B_n & \longrightarrow & \mathcal{M}_{0,1,n} & \text{push down the disk along the braid} \\ & \longrightarrow & \mathrm{Aut}(F_n) & \text{take } \pi_1. \end{array}$$

- $x_1, \ldots, x_n$ generators for F_n:

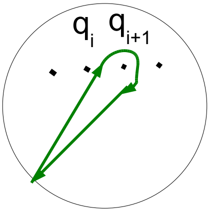
$$\sigma_i = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_{i+1} & x_{i+1}^{-1} x_i x_{i+1} & x_{i+2} & \cdots & x_n \end{pmatrix}$$

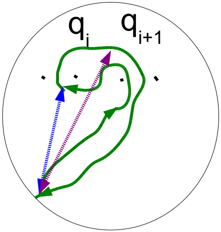
$$\sigma_i^{-1} = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i x_{i+1} x_i^{-1} & x_i & x_{i+2} & \cdots & x_n \end{pmatrix}$$

$\sigma_i \in \mathcal{M}_{0,1,n}$: Rotating inside a small disk containing q_i and q_{i+1}



$$\sigma_i(x_{i+1}) = x_{i+1}^{-1} x_i x_{i+1}$$





Artin Representation Theorem

- Under the Artin Representation $B_n \longrightarrow \mathrm{Aut}(F_n)$,
- B_n is isomorphic to the subgroup of $\mathrm{Aut}(F_n)$ consisting of **all** right automorphisms $\bar{\beta}$ on F_n such that
    - $x_i \cdot \bar{\beta}_n = A_i x_{\tau(i)} A_i^{-1}$ for $1 \leq i \leq n$, i.e. sending each generator to a conjugation of another generator,
    - $(x_1 \cdots x_n) \cdot \bar{\beta}_n = x_1 \cdots x_n$,

  where $\tau \in \Sigma_n$ and $A_i \in F_n$.
- Reference: J. S. Birman, *Braids, links, and mapping class groups*, Annals of Math. Studies **82** Princeton University Press, (1975).

Reduced Artin Representation

- $\pi_1(S^2 \setminus Q_n) = \hat{F}_n$ is the quotient group of F_n subject to the single relation: $x_1 \cdots x_n = 1$. As a group, $\hat{F}_n = F_{n-1}$ free group of rank $n-1$.
- $h \in \mathrm{Diff}^+(S_{0,1,n})$, i.e. $h\colon D^2 \setminus Q_n \cong D^2 \setminus Q_n$ such that $h|_{\partial D^2} = \mathrm{id}_{\partial D^2}$ and $h(Q_n) = Q_n$ has the canonical extension $\hat{h}\colon S^2 \setminus Q_n \cong S^2 \setminus Q_n$.
- $\Longrightarrow$ **Reduced Artin Representation**

$$\hat{h}_*\colon \hat{F}_n = \pi_1(S^2 \setminus Q_n) \xrightarrow{\cong} \hat{F}_n = \pi_1(S^2 \setminus Q_n).$$

- **Algebraically** $(x_1 \cdots x_n)\bar{\beta} = x_1 \cdots x_n \Longrightarrow$ $\hat{\Phi}\colon B_n \longrightarrow \mathrm{Aut}(\hat{F}_n)$.
- **Theorem.** $\mathrm{Ker}(\hat{\Phi}) = Z(B_n) \cong \mathbb{Z}$, the center of B_n generated by the full twist braid $(\sigma_1 \cdots \sigma_{n-1})^n$.

The group G_n

- The group G_n is defined by generators $x_1, x_2, \ldots, x_n$, and
- Relations:
  - (1) $x_1 x_2 \cdots x_n = 1$;
  - (2) All words $w = w(x_1, x_2, \ldots, x_n) \in F_n$ such that $w(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = 1$ for each $1 \leq i \leq n$.
- **Note.** Relation (2) $\Longleftrightarrow$ all iterated commutators

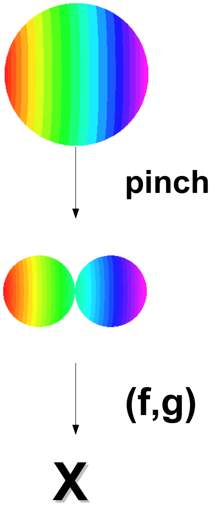
$$[x_{i_1}^{\epsilon_1}, \ldots, x_{i_t}^{\epsilon_t}] = 1$$

  if each x_i occurs at least once in $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$.
- **Theorem (Wu)** $\pi_n(S^2) \cong Z(G_n)$, the center of G_n.
- J. Wu, *Combinatorial descriptions of the homotopy groups of certain spaces*, Math. Proc. Camb. Philos. Soc. **130** (2001), 489–513.

Homotopy Group

- $\pi_n(X) := [S^n, X]$, the set of the (pointed) homotopy classes of (pointed) continuous maps from the n-sphere S^n to X.
- $\pi_0(X)$ is the set of path-connected components of X, which is not a group in general.
- fundamental group $\pi_1(X)$ is a group, but non-commutative in general.
- $\pi_n(X)$ is an abelian group for $n \geq 2$.
- **Fundamental Problem** in Algebraic Topology: Determine the homotopy groups of spheres.
- $\pi_m(S^n)$ for $m > n$ is **not yet well understood** for general m and $n \geq 2$, although many non-zero elements are known.

Product $[f] + [g]$ in $\pi_n(X)$



The braid group action on G_n

- The Artin representation $B_n \to \mathrm{Aut}(F_n)$ induces a B_n-action on G_n, i.e.

$$\begin{array}{ccc} F_n & \xrightarrow[\cong]{\bar{\beta}} & F_n \\ \downarrow & & \downarrow \\ G_n & \xrightarrow[\cong]{\bar{\beta}} & G_n. \end{array}$$

- **Sketch of Proof:** For Relation (1), $(x_1 \cdots x_n)\bar{\beta} = x_1 \cdots x_n$.

- Let $p_i\colon F_n \to F_{n-1}$ be the homomorphism:

$$p_i = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & 1 & x_i & \cdots & x_{n-1} \end{pmatrix}$$

- $w \in \text{ Relation (2) } \iff w \in \bigcap_{i=1}^{n} \mathrm{Ker}(p_i).$
- In geometry, let $Q_{n,i} = \{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots q_n\} \subseteq Q_n$ be the subset of Q_n given by deleting q_i. Let $f_i\colon D^2 \smallsetminus Q_n \to D^2 \smallsetminus Q_{n,i}$ be the inclusion. Then

$$p_i = f_{i*}\colon F_n = \pi_1(D^2 \smallsetminus Q_n) \longrightarrow F_{n-1} = \pi_1(D^2 \smallsetminus Q_{n,i}).$$

Let $h \in \mathrm{Diff}^+(S_{0,1,n})$. Then $h\colon D^2 \smallsetminus Q_n \to D^2 \smallsetminus Q_n$ with

$$\begin{array}{ccc} D^2 \smallsetminus Q_n & \xrightarrow{f_i} & D^2 \smallsetminus Q_{n,i} \\ \downarrow h & & \downarrow \hat{h} \\ D^2 \smallsetminus Q_n & \xrightarrow{f_{\tau(i)}} & D^2 \smallsetminus Q_{n,\tau(i)}, \end{array}$$

where $\hat{h}$ is the extension of h with $\hat{h}(q_i) = q_{\tau(i)}$. $\Longrightarrow$

$$\begin{array}{ccccc} \mathrm{Ker}(p_i) & \hookrightarrow & F_n = \pi_1(D^2 \smallsetminus Q_n) & \xrightarrow{p_i} & F_{n-1} = \pi_1(D^2 \smallsetminus Q_{n,i}) \\ & & \Big\downarrow \bar{\beta} = h_* & & \Big\downarrow \hat{h}_* \\ \mathrm{Ker}(p_{\tau(i)}) & \hookrightarrow & F_n = \pi_1(D^2 \smallsetminus Q_n) & \xrightarrow{p_{\tau(i)}} & F_{n-1} = \pi_1(D^2 \smallsetminus Q_{n,\tau(i)}). \end{array}$$

$$\Longrightarrow \bar{\beta}(\mathrm{Ker}(p_i)) \subseteq \mathrm{Ker}(p_{\tau(i)}) \Longrightarrow \bar{\beta}\Big(\bigcap_{i=1}^{n} \mathrm{Ker}(p_i)\Big) \subseteq \bigcap_{i=1}^{n} \mathrm{Ker}(p_i).$$

Fixed Set Theorem (Wu)

- The (reduced) Artin representation of B_n on $\hat{F}_n$ induces a B_n-action on $G(n)$.
- $\pi_n(S^2) \cong Z(G(n)) = G(n)^{P_n}$, the **fixed set** of the **pure braid group** action on $G(n)$.
- The fixed set $G(n)^{B_n} = \{x \in \pi_n(S^2) \mid 2x = 0\}$.
- J. Wu, *A braided simplicial group*, Proc. London Math. Soc. **84** (2002), 645–662.
- **Problem:** How to study $G(n)^{P_n}$ using geometric methods? There is a covering space over $S^2 \smallsetminus Q_n$ with fibre $G(n)$.

Δ-sets and simplicial sets

- A **Δ-set** means $X = \{X_n\}_{n\geq 0}$ with faces $d_i\colon X_n \to X_{n-1}$, $0 \leq i \leq n$, such that $d_i d_j = d_j d_{i+1}$ for $i \geq j$.
- A **simplicial set** means a Δ-set X with degeneracies $s_i\colon X_n \to X_{n+1}$ such that $s_j s_i = s_{i+1} s_j$ for $j \leq i$ and

$$d_j s_i = \begin{cases} s_{i-1} d_j & j < i \\ \mathrm{id} & j = i, i+1 \\ s_i d_{j-1} & j > i+1. \end{cases}$$

- One can use *deleting-doubling* for catching simplicial identities:

$$d_i\colon (x_0, \ldots, x_n) \longrightarrow (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

$$s_i\colon (x_0, \ldots, x_n) \longrightarrow (x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n)$$

singular simplicial sets

- Let $\Delta[n]=\{(x_0,x_1,\ldots,x_n)\in\mathbb{R}^{n+1}\mid x_i\geq 0,\ \sum_{i=0}^n x_i=1\}$ be the standard n-simplex.

$$d^i\colon \Delta[n-1]\longrightarrow\Delta[n]$$

$$(x_0,\ldots,x_{n-1})\mapsto(x_0,\ldots,x_{i-1},0,x_i,\ldots,x_{n-1})$$

$$s^i\colon \Delta[n+1]\longrightarrow\Delta[n]$$

$$(x_0,\ldots,x_{n+1})\mapsto(x_0,\ldots,x_{i-1},x_i+x_{i+1},x_{i+2},\ldots,x_{n+1}).$$

- Let X be any space. The **singular simplicial set** $S(X)=\{S_n(X)\}_{n\geq 0}$ with $S_n(X)=\mathrm{Map}(\Delta[n],X)$.

$$d_i=d^i_*\colon S_n(X)\longrightarrow S_{n-1}(X)\quad f\mapsto f\circ d^i$$

$$s_i=s^i_*\colon S_n(X)\longrightarrow S_{n+1}(X)\quad f\mapsto f\circ s^i.$$

Δ-group and simplicial group

- A Δ-set $X = \{X_n\}_{n\geq 0}$ is called a **Δ-group** if each X_n is a group, and each face d_i is a group homomorphism.
- A simplicial set $X = \{X_n\}_{n\geq 0}$ is called a **simplicial group** if each X_n is a group, and all faces d_i and degeneracies s_i are group homomorphisms.
- **Theorem.** Let G be an **abelian** Δ-group. Define $\partial_n = \sum_{i=0}^{n}(-1)^i d_i \colon G_n \to G_{n-1}$. Then

$$\cdots \longrightarrow G_{n+1} \xrightarrow{\partial_{n+1}} G_n \xrightarrow{\partial_n} G_n \longrightarrow \cdots$$

  is a chain complex, i.e., $\partial_n \circ \partial_{n+1} = 0$.
  $H_n(G) = \mathrm{Ker}(\partial_n)/\,\mathrm{Im}(\partial_{n+1})$.
- Proof: $\partial_n \circ \partial_{n+1} = \sum_{i=0}^{n}(-1)^i d_i \sum_{j=0}^{n+1}(-1)^j d_j = \cdots = 0$ using $d_i d_j = d_j d_{i+1}$ for $i \geq j$.

Moore Chain Complex

- **Singular homology:** Let $G = \mathbb{Z}(S(X))$. Then $H_*(X) = H_*(G)$.
- Let G be any Δ-group. Define $N_nG = \bigcap_{i=1}^{n} \mathrm{Ker}(d_i\colon G_n \to G_{n-1})$.
- **Theorem.** Let G be any Δ-group. Then

$$\cdots \longrightarrow N_{n+1}G \xrightarrow{d_0} N_nG \xrightarrow{d_0} N_{n-1}G \longrightarrow \cdots$$

  is a chain complex, i.e., $d_0|_{N_nG} \circ d_0|_{N_{n+1}G} = 1$ is trivial.
- Proof. Let $w \in N_{n+1}G$. Then, for all $i \geq 0$, $d_id_0w = d_0d_{i+1}w = d_01 = 1. \implies d_0w \in N_nG$ and $d_0(d_0w) = 1$.

Moore homotopy groups

- Let G be any Δ-group. Define $\pi_n(G)$ to be the coset $\pi_n(G) := \mathrm{Ker}(d_0|_{N_nG})/\,\mathrm{Im}(d_0(N_{n+1}G))$.
- Let G be a simplicial group. Then each $\mathrm{Im}(d_0(N_{n+1}(G)))$ is a normal subgroup and so $\pi_n(G)$ are groups. Moreover $\pi_n(G)$ abelian for $n \geq 1$.
- If G is an abelian simplicial group, then $\pi_*(G) \cong H_*(G)$.
- Let X be any path-connected space. Then there exists a simplicial group G such that $\pi_n(G) \cong \pi_{n+1}(X)$ for all $n \geq 0$.
- Reference: E. B. Curtis, *Simplicial homotopy theory*, Advances in Math. **6** (1971), 107–209.

Simplicial group model for the sphere

- $\hat{F}_{n+1} = \langle x_0, x_1, \ldots, x_n \mid x_0 x_1 \cdots x_n = 1 \rangle$
- faces and degeneracies

$$d_i = \begin{pmatrix} x_0 & x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots & x_n \\ x_0 & x_1 & \cdots & x_{i-1} & 1 & x_i & \cdots & x_{n-1} \end{pmatrix}$$

$$s_i = \begin{pmatrix} x_0 & x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots & x_n \\ x_0 & x_1 & \cdots & x_{i-1} & x_i x_{i+1} & x_{i+2} & \cdots & x_{n+1} \end{pmatrix}$$

- $\hat{F} = \{\hat{F}_{n+1}\}_{n \geq 0}$ is a simplicial group with $\pi_n(\hat{F}) \cong \pi_{n+1}(S^2)$ for all n.
- References:

  - J. Wu, *Combinatorial descriptions of the homotopy groups of certain spaces*, Math. Proc. Camb. Philos. Soc. **130** (2001), 489–513.
  - J. A. Berrick, F. R. Cohen, Y. L. Wong and J. Wu, *braids, configurations and homotopy groups*, J. Amer. Math. Soc., **19** (2006), no. 2, 265–326.

simplicial and Δ-structure on configuration spaces

- $F(M, n+1) = \{(z_0, z_1, \ldots, z_n) \in M^{n+1} \mid z_i \neq z_j \text{ for } i \neq j\}$.
- $d_i \colon (z_0, z_1, \ldots, z_n) \mapsto (z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$.
- $s_i \colon (z_0, z_1, \ldots, z_n) \mapsto (z_0, \ldots, z_{i-1}, z_i, z_i', z_{i+1}, \ldots, z_n)$ **provided** that there exists a non-where zero vector field over M, where
$$z_i' = z_i'(z_0, \ldots, z_n)$$
is **very close to but different from** z_i and is continuous on $z_0, \ldots, z_n$.

- Details: J. A. Berrick, F. R. Cohen, Y. L. Wong and J. Wu, *braids, configurations and homotopy groups*, J. Amer. Math. Soc., **19** (2006), no. 2, 265–326.

simplicial and Δ-structure on braids

- Let M be any manifold. By taking the fundamental group, $\{\pi_1(F(M,n+1))\}_{n\geq 0}$ is a Δ-group, where d_i is given by deleting the $(i+1)$st strand.
- $\{\pi_1(B(M,n+1))\}_{n\geq 0}$ is a Δ-set. (Actually so-called **crossed simplicial group**. Roughly speaking, d_i is not a homomorphism but satisfies some special properties.)
- If M has a non-where zero vector field, then $\{\pi_1(B(M,n+1))\}_{n\geq 0}$ is a crossed simplicial group.
- **Problem:** Note that $\pi_1(X)=[S^1,X]$. Let A be any pointed space. Does $\{[A,B(M,n+1)]\}_{n\geq 0}$ admit a Δ-structure (simplicial structure if M has a non-where zero vector field)? The answer is yes to $\{[A,F(M,n+1)]_{n\geq 0}\}$.
- **Reference:** J. A. Berrick, F. R. Cohen, Y. L. Wong and J. Wu, *braids, configurations and homotopy groups*, J. Amer. Math. Soc., **19** (2006), no. 2, 265–326.

Braids over the sphere

Theorem. Let $\mathcal{F}(S^2)^{\pi_1} = \{\pi_1(F(S^2, n+1))\}_{n\geq 0}$ be the Δ-group defined above. Then for each $n \geq 1$ $\pi_n(\mathcal{F}(S^2)^{\pi_1})$ is a group, and there is an isomorphism of groups

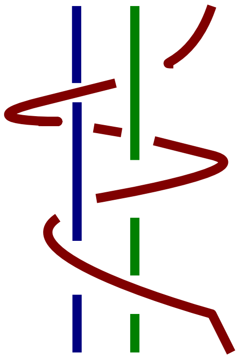
$$\pi_n(\mathcal{F}(S^2)^{\pi_1}) \cong \pi_n(S^2)$$

for $n \geq 4$.

Brunnian Braids

- An n-strand braid β over any manifold M is called **Brunnian** if it becomes a trivial braid after **deleting any one of its strands.**
- The Brunnians $\mathrm{Brun}_n(M)$ is a subgroup of the braids $B_n(M)$.

Brunnian Braids



Theorem

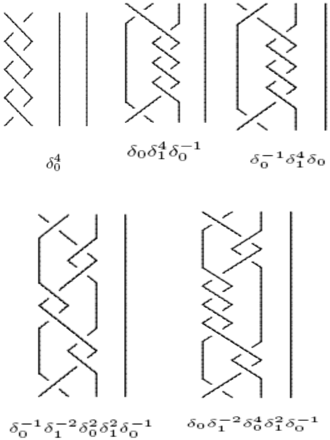
- The canonical embedding $f\colon D^2 \subseteq S^2$, as hemisphere, induces a group homomorphism $\mathrm{Brun}_n(D^2) \xrightarrow{f_*} \mathrm{Brun}_n(S^2)$.
- There is an exact sequence of groups

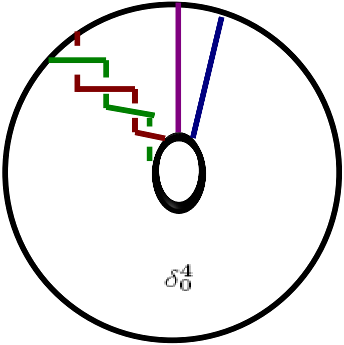
$$\mathrm{Brun}_{n+1}(S^2) \hookrightarrow \mathrm{Brun}_n(D^2) \xrightarrow{f_*} \mathrm{Brun}_n(S^2) \twoheadrightarrow \pi_{n-1}(S^2)$$

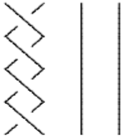
for $n \geq 5$.

- The image of $f_*\colon \mathrm{Brun}_n(D^2) \to \mathrm{Brun}_n(S^2)$ is a normal subgroup
- Both $\mathrm{Brun}_n(D^2)$ and $\mathrm{Brun}_n(S^2)$ are free groups of **infinite rank** for $n \geq 5$.

$Brunn_4(S^2)$


δ_0^4
$\delta_0\delta_1^4\delta_0^{-1}$
$\delta_0^{-1}\delta_1^4\delta_0$
$\delta_0^{-1}\delta_1^{-2}\delta_0^2\delta_1^2\delta_0^{-1}$
$\delta_0\delta_1^{-2}\delta_0^4\delta_1^2\delta_0^{-1}$





δ_0^4

Cabling

consider the free group on N letters $F_N = F_N[y_1, \cdots, y_N]$ together with elements x_i for $1 \leq i \leq N$ in P_{N+1} given by the naive “cabling" pictured in the Figure 2.1 below.

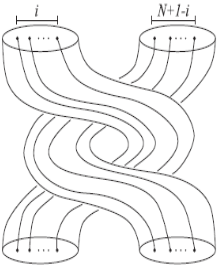


FIGURE 2.1. The braid x_i in P_{N+1}.

The braid x_1 with $N = 1 = i$ in Figure 2.1 is Artin's generator $A_{1,2}$ of P_2. The braids x_i for $1 \leq i \leq N$ in Figure 2.1 yield homomorphisms from a free group on N letters $F_N = F_N[y_1, \cdots, y_N]$ to P_{N+1}

$$\Theta_N\colon F_N[y_1, \cdots, y_N] \to P_{N+1}$$

defined on generators y_i in F_N by the formula

$$\Theta_N(y_i) = x_i.$$

Theorem

- **Theorem.** $\Theta_N\colon F_N \to P_{N+1}$ is faithful for every N, i.e., each Θ_N is a monomorphism.
- **Reference:** F. R. Cohen, and J. Wu, *On braid groups, free groups, and the loop space of the 2-sphere*, Progress in Mathematics, **215**(2003), 93-105, Birkhaüser, and *Braid groups, free groups, and the loop space of the 2-sphere*, math.AT/0409307.

Notations and Almost Brunnian braids

- $\mathrm{Brun}_k = \mathrm{Brun}_k(D^2)$ the group of k-stranded Brunnian braids.
- A group analogous to the group of Brunnian braids is the “almost Brunnian" $(k+2)$-stranded braid group

$$\mathrm{QBrun}_{k+2}(S) = \bigcap_{1 \le i \le k+1} \mathrm{Ker}(d_i \colon P_{k+2}(S) \to P_{k+1}(S)).$$

- The subgroup $\mathrm{QBrun}_{k+2}(S)$ of $P_{k+2}(S)$ consists of those braids which are trivial after deleting any one of the strands $2, 3, \cdots, k+2$, but not necessarily the first.
- In the case of $S = D^2$, the map $d_0 \colon \mathrm{QBrun}_{k+2} \to \mathrm{Brun}_{k+1}$ is a split surjection.
- The following problems are from: F. R. Cohen and J. Wu, On braid groups and homotopy groups, preprint.

Problem 1

- (1) Find natural methods to distinguish between cosets of certain braids given by homotopy groups rather than braid themselves. For example, Vassiliev invariants of pure braids distinguish all pure braids by Kohno. Are there weaker versions of Vassiliev invariants which distinguish left cosets in the braid groups given by elements in homotopy groups?

- T. Kohno, *Linear representations of braid groups and classical Yang-Baxter equations*, Cont. Math., **78**(1988), 339-363.
- T. Kohno, *Vassiliev invariants and de Rham complex on the space of knots*, in: Symplectic Geometry and Quantization, Contemp. Math., **179**(1994), Amer. Math. Soc., Providence, RI, 123-138.

Problem 2

- (2) Give combinatorial properties of the natural map $\mathrm{Brun}_{k+1}(\mathbb{R}^2) \longrightarrow \mathrm{Brun}_{k+1}(S^2)$ which provides information about the cokernel. A more precise problem is stated next. Give group theoretic reasons why the order of 2-torsion in $\pi_*(S^2)$ is bounded above by 4 and why the p-torsion for an odd prime p is bounded above by p.

Problem 3

- (3) The groups $\mathrm{QBrun}_{n+2} \cap F_{n+1}$, and $\mathrm{Brun}_{n+1} \cap F_n$ are free. Give combinatorial descriptions of the map

$$\Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2}) \to \Theta_k(F_k) \cap \mathrm{Brun}_{k+1}$$

on the level of abelianizations $H_1(\Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2}))$, and $H_1(\Theta_k(F_k) \cap \mathrm{Brun}_{k+1})$.
The Serre exact sequence for the homology of a discrete group specializes to

$$\begin{array}{ccc} \to & H_2(\pi_{k+1} S^2) & \\ \to & H_1[\Theta_k(F_k) \cap d_0(\mathrm{QBrun}_{k+2})]_{\pi_{k+1} S^2} & \\ \to & H_1[\Theta_k(F_k) \cap \mathrm{Brun}_{k+1}] & \to \pi_{k+1} S^2 \to 0 \end{array}$$

where A_π denotes the group of coinvariants of a π-module A.

Problem 4

- (4) Do Vassiliev invariants of pure braids admit descriptions as crossed homomorphisms $f\colon P_n \to M$ for some choice of module M over the group ring of the pure braid group P_n ? A related, but different example is given by the natural epimorphism
$$B_3 \to SL(2, \mathbb{Z})$$
together with crossed homomorphisms out of $SL(2, \mathbb{Z})$ which are interpreted as modular forms. Do Vassiliev invariants of braids and modular forms arise from similar contexts ? What is $H^1(B_{2g+2}; \mathbb{R}[x_1, \cdots, x_{2g}])$ where B_{2g+2} acts via the natural symplectic representation on a vector space with basis $\{x_1, \cdots, x_{2g}\}$?

Problem 5

- (5) Consider Brunnian braids Brun_k. Fix a braid γ with image in Σ_k given by a k-cycle. For any braid α in Brun_k, the braid closure of $\alpha \circ \gamma$ is a knot. Describe features of these knots or those obtained from the analogous constructions for $\Theta_k(F_{k-1}) \cap Brun_k$. Where do these “fit" in Budney’s description of the space of long knots?
- R. Budney, *The topology of knot spaces in dimension* 3, preprint, math.GT/0506523.

Thank You

Have a nice vacation!

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