NATIONAL UNIVERSITY OF SINGAPORE

Department of Mathematics

MA2108 Advanced Calculus II

2001/2002 Semester I Lecture Notes Part I

Chapter 1: Sequences

1.1. Sequences

A *sequence* is an ordered list of numbers. For example,

The order of the sequence is important. For example,

is different from ( ). An *infinite* sequence is a list which does not end. For example,

We denote by $\{a_n\}$ the sequence

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

Example

1. $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$
2. $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \cdots$
3. $1, -2, 3, -4, 5, \cdots$

Can you find a formula for each of the above sequences?

Typeset by $\mathcal{AMS}$-TEX

1.2. Limits of Sequences

Definition 1.2.1. The limit of $\{a_n\}$ is A, and is written as

$$\lim_{n\to\infty} a_n = A,$$

if for any $\epsilon > 0$, there is a natural number N such that for every $n > N$, we have

$$|a_n - A| < \epsilon.$$

Remark. 1. Some sequences do not satisfy the above. We call such sequences *divergent.*
2. Sequences which satisfy the above definition, i.e. A exists and is finite, are called *convergent* sequences.

Example 1.2.1.

Theorem 1.2.1. *If $\{a_n\}$ has a limit, then the limit is unique.*

Proof:

Theorem 1.2.2 [Squeeze or Sandwich Theorem]. *Given 3 sequences $\{a_n\}, \{b_n\}, \{c_n\}$ such that*
(i) $a_n \le b_n \le c_n$ *for every n and*
(ii) $\lim_{n\to\infty} a_n = A = \lim_{n\to\infty} c_n$,
then $\lim_{n\to\infty} b_n = A$.

Proof: Omitted.

Remark. The above theorem is still applicable if the inequality

$$a_n \le b_n \le c_n$$

is true "eventually".

Example 1.2.2.

1.3. Sequences which tend to ∞

Definition 1.3.1. $\{a_n\}$ tends to $+\infty$ if for each positive number k, there is an N such that

$$a_n > k \quad \text{for all } n > N.$$

Remark. For such sequences, we write as $a_n \to +\infty$ as $n \to \infty$ or

$$\lim_{n\to\infty} a_n = +\infty.$$

Example 1.3.1.

Theorem 1.3.2 [Reciprocal Rule]. *Consider a sequence* $\{a_n\}$.

(i) If $a_n > 0$ *for all* n *and* $\lim_{n\to\infty} \frac{1}{a_n} = 0$, *then*

$$\lim_{n\to\infty} a_n = \pm\infty.$$

(ii) If $\lim_{n\to\infty} a_n = \pm\infty$, *then* $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

Example 1.3.2.

1.4. Techniques For Computing Limits

Theorem 1.4.1. *Let f be a continuous function. Then*

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n).$$

Proof: Omitted.

Example 1.4.1.

Theorem 1.4.2 [L'Hopital's Rule]. *Suppose $a_n = f(n)$, $b_n = g(n)$. If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$, then*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

Example 1.4.2.

Theorem 1.4.3. *If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, then*

$$\begin{aligned}
\lim_{n\to\infty} (a_n + b_n) &= \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n, \\
\lim_{n\to\infty} k a_n &= k \lim_{n\to\infty} a_n, \\
\lim_{n\to\infty} a_n b_n &= \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n, \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \quad \textit{provided } b_n \neq 0, \lim_{n\to\infty} b_n \neq 0.
\end{aligned}$$

Proof: omitted.

Example 1.4.3.

Some Standard Limits

1. $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$ for any fixed $p > 0$.
2. $\lim\limits_{n\to\infty} c^n = 0$ for any fixed c where $|c| < 1$.
3. $\lim\limits_{n\to\infty} c^{\frac{1}{n}} = 1$ for any fixed $c > 0$.
4. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.
5. $\lim\limits_{n\to\infty} \dfrac{n^p}{c^n} = 0$ for any fixed p and $c > 1$.
6. $\lim\limits_{n\to\infty} \dfrac{c^n}{n!} = 0$ for any fixed c.
7. $\lim\limits_{n\to\infty} (1 + \dfrac{x}{n})^n = e^x$ for any fixed x.

Strategy: One can find the limits of many sequences from those of the standard sequences.

Example 1.4.4.

1.5. Bounded Sets and Sequences

Definition 1.5.1. A set of real numbers S is *bounded above* if there exists a finite real number M such that

$$x \leq M \qquad \forall x \in S.$$

M is called an *upper bound* of S.

Definition 1.5.2. A set of real numbers S is *bounded below* if there exists a finite real number m such that

$$m \leq x \qquad \forall x \in S.$$

m is called a *lower bound* of S.

Definition 1.5.3. A set which is both bounded above and below is called a *bounded set.*

Remark.

1. Upper bounds and lower bounds are not unique.
2. Some sets only have upper bounds but not lower bounds.
3. Some sets have only lower bounds but not upper bounds.
4. A set which is not bounded is called an *unbounded* set.

Example 1.5.1.

Theorem 1.5.2. *Every convergent sequence is bounded.*

Proof.

Corollary 1.5.3 [Test for divergence].
If $\{a_n\}$ is unbounded, then $\{a_n\}$ diverges.

Remark.

1. The converse may not be true, i.e., divergent sequence need not be unbounded.
2. The inverse may not be true, i.e., a bounded sequence may not be convergent.

Example 1.5.2.

1.6. Infinum, Supremum

Recall that any finite set of real numbers has a greatest element (maximum) and a least element (minimum).

Example 1.6.1.
$\{-2.5,\ 3.1,\ -4.4,\ 4.5,\ 5\}$

However, this property does not necessarily hold for infinite sets.

Example 1.6.2.

Definition 1.6.1. A real number M ($\neq \pm\infty$) is called the *least upper bound* or *supremum* of a set E if
(i) M is an upper bound of E, i.e., $x \leq M$ for every $x \in E$, and
(ii) if $M' < M$, then M' is not an upper bound of E (i.e., there is an $x \in E$ such that $M' < x$).
We write $M = \sup E$.

Remark.
(i) $\sup E$ is unique whenever it exists.
(ii) The main difference between $\sup E$ and $\max E$ is that $\sup E$ may not be an element of E, whereas $\max E$ must be an element of E if it does exist).
(iii) If E has a maximum, then $\sup E = \max E$.

Example 1.6.3.

Definition 1.6.2. A real number m ($\neq \pm\infty$) is called the *greatest lower bound* or *infimum* of a set E if
(i) m is a lower bound of E, i.e., $m \leq x$ for every $x \in E$, and
(ii) if $m' > m$, then m' is not a lower bound of E (i.e., there exists an $x \in E$ such that $x < m'$).
We write $m = \inf E$.

Remark.
(i) $\inf E$ is unique whenever it exists.
(ii) The main difference between $\inf E$ and $\min E$ is that $\inf E$ may not be an element of E, whereas $\min E$ must be an element of E if it does exist.
(iii) If E has a minimum, then $\inf E = \min E$.

Example 1.6.4.

An important property of the set of real numbers is the following

1.6.1 (Completeness Axiom of $\mathbb{R}$).
(i) *If E is bounded above, then* $\sup E$ *exists.*
(ii) *If E is bounded below, then* $\inf E$ *exists.*

Recall that a set E is bounded if and only if it is bounded above and bounded below. Thus the Completeness Axiom leads to

Corollary 1.6.2. *If E is bounded, then both* $\sup E$ *and* $\inf E$ *exist.*

1.7 Monotone Sequences

Definition 1.7.1. $\{a_n\}$ is called *monotone increasing (decreasing)* if $a_n \leq \ (\geq)\ a_{n+1}$ for every n.

Example 1.7.1.

Proposition 1.7.1. *A monotone increasing (decreasing) sequence is bounded below (above).*

Proof.

Theorem 1.7.2 [Monotone Convergence Theorem]. *A bounded monotone sequence* $\{a_n\}$ *is convergent, and the limit is*
$\lim\limits_{n\to\infty} a_n = \sup\limits_n a_n$ *if the sequence is monotone increasing,*
$\lim\limits_{n\to\infty} a_n = \inf\limits_n a_n$ *if the sequence is monotone decreasing.*

Example 1.7.2.

Corollary 1.7.3. *If* $\{a_n\}$ *is monotone increasing (decreasing), then either*
(i) $\{a_n\}$ *is convergent or*
(ii) $\lim\limits_{n\to\infty} a_n = +\infty(-\infty)$.

Proof.

1.8. Subsequences

Example.

1. Find the following subsequences of $\{a_n\} = \{1, -1, 1, -1, 1, -1, \cdots\}$.

$\{a_{2n-1}\} =$

$\{a_{2n}\} =$.

In general, subsequences of $\{a_n\}$ are of the form $\{a_{n_k}\}$, $k = 1, 2, 3, ...$, with

$$n_1 < n_2 < n_3 < \cdots .$$

Theorem 1.8.1. *Suppose* $\lim\limits_{n\to\infty} a_n = A$. *Then every subsequence of* $\{a_n\}$ *also converges to* A*, ie*

$$\lim_{k\to\infty} a_{n_k} = A.$$

Theorem 1.8.2. *Suppose that* $\{a_n\}$ *has 2 subsequences which converges to different limits. Then* $\{a_n\}$ *is divergent.*

Example 1.8.1.

1.9. The $\limsup$ and $\liminf$ of a sequence

Reference: W. Wade, *An introduction to analysis*, Prentice Hall, 1995.

Given a sequence $\{a_n\}$, we can form another sequence $\{b_n\}$ given by

$$\begin{aligned} b_n &= \sup_{k\geq n} a_k \\ &= \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\}. \end{aligned}$$

Example 1.9.1.

Proposition 1.9.1.
For any sequence $\{a_n\}$, the associated sequence $\{b_n\} = \{\sup_{k\geq n} a_k\}$ is always monotone decreasing.

Proof. For each n,

$$\begin{aligned} b_n &= \sup\{a_n, a_{n+1}, a_{n+2}, \cdots\} \\ &\geq \sup\{a_{n+1}, a_{n+2}, \cdots\} = b_{n+1}. \end{aligned}$$

Definition 1.9.1. The *limit superior* of $\{a_n\}$, denoted by $\limsup a_n$ or $\limsup\limits_{n\to\infty} a_n$ or $\overline{\lim\limits_{n\to\infty}}\, a_n$ is defined to be $\lim\limits_{n\to\infty} b_n$, i.e.

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \sup_{k\geq n} a_k.$$

Example 1.9.2.

Proposition 1.9.2. *Given any $\{a_n\}$, $\limsup a_n$ always exists (either finite, $+\infty$ or $-\infty$).*

Proof. If $\{a_n\}$ is not bounded above, then each b_n is $+\infty$, and thus $\limsup a_n = \lim\limits_{n\to\infty} b_n = +\infty$. If $\{a_n\}$ is bounded above, then each b_n is finite. Since $\{b_n\}$ is monotone decreasing, by Corollary 1.7.3, $\{a_n\}$ convertes (to a finite limit), or $\lim_{n\to\infty} b_n = -\infty$.

Similarly, given any sequence sequence $\{a_n\}$, we can form another sequence $\{c_n\}$ given by

$$\begin{aligned} c_n &= \inf_{k \geq n} a_k \\ &= \inf\{a_n, a_{n+1}, a_{n+2}, \cdots\}. \end{aligned}$$

Definition 1.9.2. The *limit inferior* of $\{a_n\}$, denoted by $\liminf a_n$ or $\liminf_{n\to\infty} a_n$ or $\varliminf_{n\to\infty} a_n$ is defined to be $\lim_{n\to\infty} c_n$, i.e.

$$\varliminf_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} \inf_{k\geq n} a_k.$$

Example 1.9.3.

Proposition 1.9.3. *(i) As in Proposition 1.9.1, for any given sequence $\{a_n\}$, the associated sequence $\{c_n\} = \{\inf_{k\geq n} a_k\}$ is always monotone increasing.*
(ii) As in Proposition 1.9.2, for any given $\{a_n\}$, $\liminf a_n$ always exists (either finite, $+\infty$ or $-\infty$).

Remark.
We always have $\varliminf_{n\to\infty} a_n \leq \varlimsup_{n\to\infty} a_n$.

Proposition 1.9.4. *(i) If $\varlimsup_{n\to\infty} a_n = B$, then given $\epsilon > 0$, there exists N such that $a_n < B + \epsilon$ for all $n > N$.*
(ii) $\varliminf_{n\to\infty} a_n = C$, then given $\epsilon > 0$, there exists N such that $a_n > C - \epsilon$ for all $n > N$.

Proof. (i) Since $\varlimsup_{n\to\infty} a_n = B$, given any $\epsilon > 0$, there exists N such that for all $n > N$,

$$|b_n - B| < \epsilon \Rightarrow b_n < B + \epsilon \Rightarrow \sup\{a_n, a_{n+1}, \cdots\} < B + \epsilon,$$

i.e. $a_n, a_n + 1, \cdots < B + \epsilon$ for all $n > N$. Proof of (ii) is similar.

Remark. Roughly speaking, Proposition 1.9.4 says that for any sequence $\{a_n\}$, the a_n's are eventually not much smaller than $\varliminf_{n\to\infty} a_n$ and not much bigger than $\varlimsup_{n\to\infty} a_n$.

Proposition 1.9.5. *If* $\varliminf_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = A$ *(finite), then* $\{a_n\}$ *converges and* $\lim_{n\to\infty} a_n = A$*.*

Proof. Use Proposition 1.9.4 (Exercise).

1.10. Cauchy Sequences

Definition 1.10.1. $\{a_n\}$ is called a *Cauchy sequence* if given any $\epsilon > 0$, there exists a natural number N such that for all m, $n > N$, we have

$$|a_n - a_m| < \epsilon.$$

Remark.
Roughly speaking, a sequence is Cauchy if the width of its tail $\rightarrow$ 0 as $n \rightarrow \infty$.

Proposition 1.10.1. *Every Cauchy sequence is bounded.*

Theorem 1.10.2 [Cauchy's criterion].
A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Proof.

Chapter 2: Series

2.1. Series

The expression

$$a_1 + a_2 + a_3 + \cdots$$

written alternatively as $\sum_{k=1}^{\infty} a_k$ is called an *infinite series.*

Example 2.1.1.

Definition 2.1.1. Given a series $\sum_{k=1}^{\infty} a_k$, its n^{th} *partial sum* S_n is given by

$$S_n = \sum_{k=1}^{n} a_k =$$

The sequence $\{S_n\}$ is called the *sequence of partial sums* of the series $\sum_{k=1}^{\infty} a_k$.

Example 2.1.2.

Definition 2.1.2. Consider the sequence of partial sums $\{S_n\}$ of the series $\sum_{k=1}^{\infty} a_k$. If this sequence converges to a number S, we say that the series $\sum_{k=1}^{\infty} a_k$ converges to S and write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S.$$

If $\{S_n\}$ diverges, then we say $\sum\limits_{k=1}^{\infty} a_n$ *diverges.*

Example 2.1.3.

Remark. One always has

(i) $\sum\limits_{k=1}^{\infty}(a_k + b_k) = \sum\limits_{k=1}^{\infty} a_k + \sum\limits_{k=1}^{\infty} b_k.$

(ii) $\sum\limits_{k=1}^{\infty} ca_k = c\sum\limits_{k=1}^{\infty} a_k.$

Example 2.1.4.

Theorem 2.1.1.

If $\sum\limits_{k=1}^{\infty} a_k$ *converges, then* $\lim\limits_{k\to\infty} a_k = 0.$

Corollary 2.1.2 [n-th term test for divergence].

If $\lim\limits_{n\to\infty} a_n \neq 0$ *(or does not exist), then* $\sum\limits_{n=1}^{\infty} a_n$ *diverges.*

Example 2.1.5.

Remark. The divergence test is a "one-way" test, i.e., $\lim\limits_{n\to\infty} a_n = 0$ does NOT imply $\sum\limits_{n=1}^{\infty} a_n$ converges.

Example 2.1.6.

1. The harmonic series.

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\\ \lim_{k\to\infty}a_k &= \lim_{k\to\infty}\frac{1}{k}=0.\\ S_n &= \sum_{k=1}^{n}\frac{1}{k}\\ S_{2^n} &=\end{aligned}$$

Hence, the subsequence $\{S_{2^n}\}$ is unbounded. So $\{S_n\}$ is also unbounded. Hence, $\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent

2.2. Tests for Positive Series

Goal: Given a series, we want to be able to test if it converges or not.

Definition 2.2.1. A series $\sum_{k=1}^{\infty} a_k$ is called a *positive series* if every term a_k is positive.

We first develop tests for positive series.

Proposition 2.2.2. *For a positive series* $\sum_{k=1}^{\infty} a_k$*, the sequence of partial sums* $\{S_n\}$ *is monotone increasing.*

Proof. This is so since $S_{n+1} - S_n = a_{n+1} > 0$ for each n.

Corollary 2.2.3.
(i) If $\{S_n\}$ *is bounded for a positive series, then the series converges.*
(ii) If $\{S_n\}$ *is not bounded from above, then the series diverges.*

Theorem 2.2.4 [Comparison Test].
Consider 2 positive series $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$*. Suppose that eventually* $0 \leq a_k \leq b_k$*.*
(i) If $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*
(ii) If $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Proof.

Example 2.2.1.

Remark.

1. Suppose $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges. Then NO conclusion can be drawn.
2. Similarly, suppose $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} a_n$ converges. Then NO conclusion can be drawn.

Example 2.2.2.

Corollary 2.2.5 [Limit Comparison Test].
If $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are (eventually) positive, and*

$$\lim_{k\to\infty} \frac{b_k}{a_k} = L(\neq 0, \neq \infty),$$

then either both series converge or both series diverge.

Proof.

2 standard series used in comparison and limit comparison tests.

1. The Geometric Series:

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \text{converges} & \text{if } |r| < 1, \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

2. The p-series: for a fixed p,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

To be proved in the section on Integral Test.

Example 2.2.3.

Corollary 2.2.6 [Limit Comparison Test for "$a_n << b_n$"]
Suppose $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are eventually positive, and* $\lim_{k\to\infty} \frac{b_k}{a_k} = 0.$

(i) *If* $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_n$ *converges.*

(i) *If* $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_n$ *diverges.*

Proof. $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ implies for every $\epsilon > 0$, there is an N such that

$$|\frac{a_k}{b_k} - 0| < \epsilon \quad \forall k > N.$$

We choose $\epsilon = 1$. Then the above inequality is

$$a_k < b_k \quad \forall k > N.$$

We get the result by applying the comparison test.

Remark. 1. If $\lim\limits_{k\to\infty} \dfrac{a_k}{b_k} = +\infty$, then $\lim\limits_{k\to\infty} \dfrac{b_k}{a_k} = 0$. We may then apply Corollary 2.2.6.

Example 2.2.4.

Theorem 2.2.7. [Integral Test].
Let $f : [1, \infty) \to \mathbb{R}$ be a positive, continuous and monotone decreasing function. Suppose we have a series $\sum\limits_{k=1}^{\infty} a_k$ such that $a_k = f(k)$, then the series $\sum\limits_{k=1}^{\infty} a_k$ and the integral $\int_1^\infty f(x)dx$ either both converge or both diverge.

Review on improper integrals

Let $f(x) : [1, \infty) \to [0, \infty)$ be a continuous function. Then one defines the improper integral

$$\begin{aligned}\int_1^\infty f(x)\,dx &= \lim_{n\to\infty} \int_1^n f(x)\,dx \\ &= \text{area under } f(x) \text{ over } [1, \infty).\end{aligned}$$

Here we say that $\displaystyle\int_1^\infty f(x)\,dx$ converges if the limit $\displaystyle\lim_{n\to\infty} \int_1^n f(x)\,dx$ exists (finite), i.e., the area under $f(x)$ over $[1, \infty)$ is finite.

We also say that $\displaystyle\int_1^\infty f(x)\,dx$ diverges if the limit $\displaystyle\lim_{n\to\infty} \int_1^n f(x)\,dx$ does not exist.

The improper integrals can be defined similarly for integrals over other intervals such as $[0, \infty)$, $(-\infty, 0]$.

Proof of the Integral test: Let $f(x)$ be a positive, continuous, monotone decreasing function such that $a_n = f(n)$ for all n.

From the graph, we see that

$$\text{area of the rectangles } \le \text{area under } f(x) \text{ over } [1,n], \text{ i.e.,}$$
$$\sum_{k=2}^{n} f(k) \le \int_1^n f(x)\,dx \le \int_1^\infty f(x)\,dx.$$

Thus, if $\int_1^\infty f(x)\,dx < \infty$, then

$$\sum_{k=2}^{n} a_k = \sum_{k=2}^{n} f(k) \le \int_1^\infty f(x)\,dx < \infty,$$

i.e., for all n, $\sum_{k=2}^{n} a_k$ is bounded above by the finite number $\int_1^\infty f(x)\,dx$. Since we also have $a_k \ge 0$, it follows from Corollary 2.2.3 that $\sum_{k=2}^{\infty} a_k$ converges, and thus $\sum_{k=1}^{\infty} a_k$ also converges.

Next we consider the following graph:

From the graph, it is easy to see that

$$\text{area under the rectangles} \ge \text{area under } f(x) \text{ over } [1,n], \text{ i.e.,}$$
$$\sum_{k=1}^{n-1} f(k) \ge \int_1^n f(x)\,dx.$$

Thus, if $\sum_{k=1}^{\infty} a_k < \infty$, then

$$\infty > \sum_{k=1}^{\infty} a_k \geq \sum_{k=1}^{n-1} a_k = \sum_{k=1}^{n-1} f(k) \geq \int_1^n f(x)\,dx,$$

i.e., $\int_1^n f(x)\,dx$ is bounded above by the finite number $\sum_{k=1}^{\infty} a_k < \infty$. Letting $n \to \infty$, it follows that we have

$$\int_1^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k < \infty.$$

In conclusion, we have $\sum_{k=1}^{\infty} a_k$ converges iff $\int_1^{\infty} f(x)\,dx$ converges, which also means that $\sum_{k=1}^{\infty} a_k$ diverges iff $\int_1^{\infty} f(x)\,dx$ diverges.

Example 2.2.5.

Theorem 2.2.8 [Ratio Test].

Consider the positive series $\sum_{n=1}^{\infty} a_n$. *Suppose*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell. \tag{1}$$

(i) If $0 \leq \ell < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) If $1 < \ell \leq \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

(iii) If $\ell = 1$, *then the test is inconclusive.*

Proof. We will prove (i) and (ii). Given any $\epsilon > 0$, it follows from (1) that there exists N such that for all $n > N$,

$$\left| \frac{a_{n+1}}{a_n} - \ell \right| < \epsilon$$

$$\ell - \epsilon < \frac{a_{n+1}}{a_n} < \ell + \epsilon.$$

By repeating using the above inequalties, it follows that for all $m > 0$,

$$a_{N+1}(\ell-\epsilon)^m < a_{N+1+m} < a_{N+1}(\ell+\epsilon)^m. \tag{2}$$

(i) If $\ell < 1$, choose $\epsilon > 0$ such that $\ell + \epsilon < 1$, then $\displaystyle\sum_{m=1}^{\infty} a_{N+1}(\ell+\epsilon)^m$ converges (since it is a geometric series with common ratio satisfying $|r| = \ell + \epsilon < 1$). Together with RHS of (2), it follows from the comparison test that $\displaystyle\sum_{m=1}^{\infty} a_{N+1+m}$ converges, and thus $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\ell > 1$, choose $\epsilon > 0$ such that $\ell - \epsilon > 1$, then by the LHS of (2), we have, for all $m > 0$,

$$a_{N+1+m} \geq a_{N+1}(\ell-\epsilon)^m > a_{N+1} > 0.$$

In particular, $\displaystyle\lim_{n\to\infty} a_n \neq 0$ or does not exist. By the n-th term test for divergence, $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Example 2.2.6.

Theorem 2.2.9 [Root Test].

Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$ *with each* $a_n \geq 0$, *and let*

$$\ell = \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}. \tag{1}$$

(i) If $0 \leq \ell < 1$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) If $1 < \ell \leq \infty$, *then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges.*

(iii) If $\ell = 1$, *then the test is inconclusive.*

Proof. We will prove (i) and (ii).

(i) Suppose that $\ell < 1$. Then for all given $\epsilon > 0$, it follows from (1) and Proposition 1.10.4 that there exists an N such that $\sqrt[n]{a_n} < \ell + \epsilon$ for all $n > N$. Now choose $\epsilon > 0$ s.t. $\ell + \epsilon < 1$. Then

$$0 \leq a_n < (\ell+\epsilon)^n \quad \text{for all } n > N. \tag{2}$$

Since $\sum_{n=1}^{\infty}(\ell+\epsilon)^n$ converges (as it is a geometric series with common ratio satisfying $|r| = \ell + \epsilon < 1$), it follows from (2) and the comparison test that $\sum_{n=1}^{\infty} a_n$ converges.

(ii) We are going to prove (ii) by contradiction. Given that $\ell > 1$. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then by the n-th term test for divergence, we have $\lim_{n\to\infty} a_n = 0$. In particular, there exists N such that $0 \leq a_n < 1$ for all $n > N$. Hence $\sqrt[n]{a_n} < 1$ for all $n > N$, and it follows that we must have $\ell \leq 1$, which is a contradiction. Hence $\sum_{n=1}^{\infty} a_n$ diverges.

Corollary 2.2.10 [Simplified Root Test].

Consider the series $\sum_{n=1}^{\infty} a_n$ *with each* $a_n \geq 0$*. Suppose that* $\lim_{n\to\infty} \sqrt[n]{a_n} = \ell$*.*

(i) If $0 \leq \ell < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) If $1 < \ell \leq \infty$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

(iii) If $\ell = 1$*, then the test is inconclusive.*

Proof. We will prove (i) and (ii). Recall from Tutorial 3 that if $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists, then $\lim_{n\to\infty} \sqrt[n]{a_n} = \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}$. Then the Corollary follows from Theorem 2.2.9.

Example 2.2.7.

2.3. Series with both +ve and -ve terms

Example 2.3.1.

1. $2+4-6-8+10+12-14-16+\cdots$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$
3. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2}=-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\cdots$

Definition 2.3.1.

An *alternating series* is of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4+\cdots\ , \quad \text{or}$$

$$\sum_{n=1}^{\infty}(-1)^{n}a_n=-a_1+a_2-a_3+a_4-\cdots$$

with each $a_n>0$.

Example 2.3.2.

Theorem 2.3.2 [The Alternating Series test]
Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *(or* $\sum_{n=1}^{\infty}(-1)^{n}a_n$*) be an alternating series. Suppose that*
(i) $a_n > 0$ *for all* n*,*
(ii) a_n *is monotone decreasing*
(i.e., $a_n \geq a_{n+1}$ *for all* n*), and*
(iii) $\lim\limits_{n\to 0} a_n = 0$.
Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *(resp.* $\sum_{n=1}^{\infty}(-1)^{n}a_n$*) is convergent.*

Proof. (sketch) We look at the partial sums $\{S_n\}$ of the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$. Now,

$$\begin{aligned} S_{2n} &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n} \\ &\leq a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n} + a_{2n+1} - a_{2n+2} = S_{2n+2}. \end{aligned}$$

Thus $\{S_{2n}\}$ is monotone increasing. Also,

$$S_{2n} = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \cdots + a_{2n-1} - a_{2n} \leq a_1.$$

Thus $\{S_{2n}\}$ is bounded above by a_1. Then by Monotone Convergence theorem, $\{S_{2n}\}$ is convergent, and write $\lim\limits_{n\to\infty} S_{2n} = S$. We also have

$$S_{2n+1} = S_{2n} + a_{2n+1}.$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = S + 0 = S.$$

Since $\lim\limits_{n\to\infty} S_{2n} = \lim\limits_{n\to\infty} S_{2n+1} = S$, it follows that $\lim\limits_{n\to\infty} S_n = S$ (convergent), and thus $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

Example 2.3.3.

2.4. Absolute and Conditional Convergence

Theorem 2.4.1.
If $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof.

Example 2.4.1.

Definition 2.4.2.
We say that the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges.

Remark.
If you are testing for absolute convergence, all the techniques for the +ve series are applicable.

Example 2.4.2.

Corollary 2.4.3.
Every absolutely convergent series is convergent.

Proof. $\sum_{n=1}^{\infty} a_n$ absolutely convergent implies $\sum_{n=1}^{\infty} |a_n|$ converges. By Theorem 2.4.1, $\sum_{n=1}^{\infty} a_n$ converges.

Q: Is the converse of the Corollary true? I.e., if a series is convergent, will it be absolutely convergent?
A: No, it is not necessarily true.

Example 2.4.3.

Definition 2.4.4.
$\sum_{n=1}^{\infty} a_n$ is said to be *conditionally convergent* if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example 2.4.4.

Theorem 2.4.5.
Every series is either absolutely convergent, conditionally convergent or divergent.

Example 2.4.4.

2.5. Remarks on the various tests for convergence/divergence of series

1. **n-th term test for divergence:**

- a test for divergence ONLY, and it works for series with positive and negative terms, e.g. $\sum_{n=1}^{\infty}(-1)^n$.

2. **Comparison test/Limit Comparison test:**

- when applying these tests, one usually compares the given series with a geometric series or a p-series.
- generally works for series which look like the geometric series or the p-series, e.g. $\sum_{n=1}^{\infty}\frac{2+(-1)^n}{4^n}$, $\sum_{n=1}^{\infty}\frac{2^{\frac{1}{n}}}{n^2}$.
- when an oscillating factor/term appears, e.g. $\sum_{n=1}^{\infty}\frac{2+(-1)^n}{3^n}$, try the Comparison test rather than the Limit Comparison test.

3. **Integral test:** e.g. $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$.

4. **Ratio test:**

- generally works for series which look like the geometric series, series with $n!$, and certain series defined recursively, e.g. $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$, $\sum_{n=1}^{\infty}\frac{(2n)!}{4^n\cdot n!}$, $\sum_{n=1}^{\infty}a_n$, where $a_1=1$, $a_n=(\frac{1}{2}+\frac{1}{n})a_{n-1}$, $n=2,3,\cdots$.

5. **(Simplified) Root test:**

- generally works for series where a_n involves a high power such as the n-th power, e.g. $\sum_{n=1}^{\infty}\frac{n}{3^n}$, $\sum_{n=1}^{\infty}2^n\Big(1-\frac{1}{n}\Big)^{n^2}$.

6. **Alternating Series test:** - works for alternating series only, e.g. $\sum_{n=2}^{\infty}(-1)^n\frac{\ln n}{n}$.

Remark.

In general, Tests 2 - 5 works only for $\sum_{n=1}^{\infty} a_n$, where $a_n \geq 0$.