MA2108 **Midterm** **25 September 2001**
Professor J. Wu

Time allowed: 1.5 hours

Tutorial Group:(circle one)

Tuesday 12-1	Thursday 12-1	Thursday 1-2	Friday 4-5
Tuesday 1-2	Friday 9-10	Friday 10-11	Friday 11-12

ID number:______________________________
Name:______________________________
Signature:______________________________

ANSWERS to #1 − 8:

1. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
2. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
3. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
4. A ◯ B ◯ C ◯ D ◯ E ◯ F ◯
5. A ◯ B ◯
6. A ◯ B ◯
7. A ◯ B ◯ C ◯
8. A ◯ B ◯ C ◯

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Problem #	Your Grades
1 (8 points)	
2 (8 points)	
3 (8 points)	
4 (8 points)	
5 (8 points)	
6 (8 points)	
7 (8 points)	
8 (8 points)	
9 (18 points)	
10 (18 points)	
total (100 points)	

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1. Determine limit of the sequence

$$\left\{\ln\frac{2n+1}{5+2n}+\cos\frac{n^2\pi+1}{4n+2n^2+2}\right\}.$$

Answer:
(A). -1 (B). 0 (C). 1. (D). $\ln 2$. (E). $+\infty$ (F). the limit does not exist.

Solution.

$$\lim_{n\to\infty}\left(\ln\frac{2n+1}{5+2n}+\cos\frac{n^2\pi+1}{4n+2n^2+2}\right)=\ln\lim_{n\to\infty}\frac{2+\frac{1}{n}}{\frac{5}{n}+2}+\cos\lim_{n\to\infty}\frac{\pi+\frac{1}{n^2}}{\frac{4}{n}+2+\frac{2}{n^2}}=\ln 1+\cos\frac{\pi}{2}.$$

□

2. The limit of the sequence $\left\{\left(1-\frac{3}{n+1}\right)^{2n}\right\}$ is

(A). 0. (B). divergent. (C). e^{-1}. (D). e^2. (E). e^{-3}. (F). e^{-6}.

Solution.

$$\lim_{n\to\infty}\left(1-\frac{3}{n+1}\right)^{2n}=\frac{\lim_{n\to\infty}\left[\left(1+\frac{-3}{n+1}\right)^{n+1}\right]^2}{\lim_{n\to\infty}\left(1+\frac{-3}{n+1}\right)^2}=\frac{(e^{-3})^2}{1}=e^{-6}.$$

□

3. The limit of the sequence $\left\{\dfrac{1}{\sqrt{n}(\sqrt{2+n}-\sqrt{n})}\right\}$ is

(A). -2. (B). -1. (C). 0. (D). 1. (E). 2. (F). ∞.

Solution.

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}(\sqrt{2+n}-\sqrt{n})}=\lim_{n\to\infty}\frac{\sqrt{2+n}+\sqrt{n}}{\sqrt{n}\cdot(\sqrt{2+n}-\sqrt{n})\cdot(\sqrt{2+n}+\sqrt{n})}$$

$$=\lim_{n\to\infty}\frac{\sqrt{2+n}+\sqrt{n}}{\sqrt{n}\cdot 2}=\lim_{n\to\infty}\frac{\sqrt{\frac{2}{n}+1}+1}{2}=\frac{2}{2}=1.$$

□

4. The limit of the sequence $\left\{\dfrac{(n+2)!+n^3}{n^2(3^n+n!)}\right\}$ is

(A). 0. (B). $+\infty$. (C). 1. (D). 2. (E). 3. (F). 6.

Proof.

$$\lim_{n\to\infty}\frac{(n+2)!+n^3}{n^2(3^n+n!)}=\lim_{n\to\infty}\frac{\frac{(n+2)(n+1)}{n^2}+\frac{n^3}{n!}}{\frac{3^n}{n!}+1}=\lim_{n\to\infty}\frac{(1+\frac{2}{n})(1+\frac{1}{n})+\frac{n^3}{n!}}{\frac{3^n}{n!}+1}=\frac{1+0}{0+1}=1.$$

□

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Determine the absolute convergence, conditional convergence or divergence of the following series.

5. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$. Answer: (A). convergence. (B). divergence.

Solution. Since $\frac{1}{2n-1} \geq \frac{1}{2n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the p-series, the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ is divergent by the comparison test . $\square$

6. $\sum_{n=1}^{\infty} \frac{3^n + n^3}{4^n + n^4}$ Answer: (A) convergence. (B). divergence.

Solution. Let $a_n = \frac{3^n + n^3}{4^n + n^4}$ and let $b_n = \frac{3^n}{4^n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3^n + n^3) \cdot 4^n}{(4^n + n^4) \cdot 3^n} = \lim_{n\to\infty} \frac{1 + \frac{n^3}{4^n}}{1 + \frac{n^4}{3^n}} = \frac{1+0}{1+0} = 1$$

and the series $\sum_{n=1}^{\infty} b_n$ is convergent by the geometric series, the series $\sum_{n=1}^{\infty} a_n$ is convergent by the limit comparison test. $\square$

7. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}+2}$

Answer: (A). absolute convergence. (B). conditional convergence. (C). divergence.

Solution. Since $a_n = \frac{1}{\sqrt{n}+2}$ is positive, monotone decreasing and $\lim_{n\to\infty}\frac{1}{\sqrt{n}+2} = 0$, the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}+2}$ is convergent by the alternating series test.

Since $\frac{1}{\sqrt{n}+2} \geq \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}}$ for $n \geq 4$ and the series $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent by the p-series, the series

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{\sqrt{n}+2}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+2}$$

is divergent by the comparison test.

Thus the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}+2}$ is conditionally convergent. □

8. $\sum_{n=1}^{\infty}\frac{(-1)^n \sin n}{n(\ln n+1)^2}$.

Answer: (A). absolute convergence. (B). conditional convergence. (C). divergence.

Solution. Observe that $\left|\frac{(-1)^n \sin n}{n(\ln n+1)^2}\right| \leq \frac{1}{n(\ln n+1)^2}$. Let $f(x) = \frac{1}{x(\ln x+1)^2}$ for $x \geq 1$. Then $f(x)$ is positive, continuous and monotone decreasing on the interval $[1, +\infty)$. Since

$$\int_1^{\infty}\frac{1}{x(\ln x+1)^2}dx = \int_1^{\infty}\frac{1}{y^2}dy = -\frac{1}{y}\bigg|_1^{\infty} = 1,$$

where $y = \ln x + 1$, is convergent, the series $\sum_{n=1}^{\infty}\frac{1}{n(\ln n+1)^2}$ is convergent by the integral test. Thus the series $\sum_{n=1}^{\infty}\left|\frac{(-1)^n \sin n}{n(\ln n+1)^2}\right|$ is convergent by the comparison test and so the series $\sum_{n=1}^{\infty}\frac{(-1)^n \sin n}{n(\ln n+1)^2}$ is absolutely convergent. □

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9. (a) Let A and B be nonempty bounded sets of real numbers. Suppose that A is a subset of B. Show that $\sup A \leq \sup B$.

This is a tutorial question.

(b) Consider the sequence $\{a_n\}$ given by

$$a_1 = \sqrt{3}, \qquad a_n = \sqrt{3a_{n-1}} \quad n = 2, 3, 4, \cdots .$$

Show that $\{a_n\}$ converges, and find the limit.

Solution. We first show that $0 \leq a_n \leq 3$ by induction on n. Clearly $0 \leq a_1 \leq 3$. Suppose that $0 \leq a_{n-1} \leq 3$ with $n \geq 2$. Then $0 \leq a_n = \sqrt{3a_{n-1}} \leq \sqrt{3 \cdot 3} = 3$. The induction is finished and so $0 \leq a_n \leq 3$ for all n. Now since $a_n = \sqrt{3 \cdot a_{n-1}} \geq \sqrt{a_{n-1} \cdot a_{n-1}} = a_{n-1}$, the sequence $\{a_n\}$ is monotone increasing and bounded. Thus the limit $\lim_{n\to\infty} a_n$ exists.

Let $A = \lim_{n\to\infty} a_n$. Then

$$A = \lim_{n\to\infty} a_n = \sqrt{\lim_{n\to\infty} 3 \cdot a_{n-1}} = \sqrt{3A}$$

and so $A = 0$ or 3. Since $A = \lim_{n\to\infty} a_n \geq \lim_{n\to\infty} a_1 = \lim_{n\to\infty} \sqrt{3} = \sqrt{3}$, we have $A \neq 0$ and so $A = 3$. $\square$

10. (a) Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0$. Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent. Show that the series $\sum_{n=1}^{\infty} a_n^2$ is also convergent.

This is a tutorial question.

(b) Determine the domain of the function $f(x)$ defined by $f(x) = \sum_{n=1}^{\infty} n(2x-1)^n$.

$0 < x < 1$.

Solution. Let $a_n = n(2x-1)^n$. Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)|2x-1|^{n+1}}{n|2x-1|^n} = \lim_{n\to\infty} (1+\frac{1}{n})|2x-1| = |2x-1|.$$

When $l = |2x-1| < 1$, that is, $-1 < 2x-1 < 1$ or $0 < x < 1$, $f(x)$ is defined as the series is convergent by the ration test for general series.

When $l = |2x-1| > 1$, that is, $x < 0$ or $x > 2$, $f(x)$ is not defined as the series is divergent by the ration test for general series.

When $x = 0$, the series $\sum_{n=1}^{\infty} n(-1)^n$ is divergent by the divergence test because the unbounded sequence $\{n(-1)^n\}$ does not tend to zero.

When $x = 1$, the series $\sum_{n=1}^{\infty} n$ is divergent by the divergence test because the unbounded sequence $\{n\}$ does not tend to zero.

Thus the domain of $f(x)$ is the open interval $(0, 1)$.

□