Solutions to the Final Exam of MA 2108, June 2000

1 (i).

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}(\sqrt{n+8}-\sqrt{n})} = \lim_{n\to\infty} \frac{\sqrt{n+8}+\sqrt{n}}{\sqrt{n+1}(\sqrt{n+8}-\sqrt{n})(\sqrt{n+8}+\sqrt{n})}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+8}+\sqrt{n}}{8\sqrt{n+1}} = \lim_{n\to\infty} \frac{\sqrt{1+8/n}+1}{8\sqrt{1+1/n}} = \frac{2}{8} = \frac{1}{4}.$$

1 (ii).

$$\lim_{n\to\infty} \left(\frac{n}{n+3}\right)^{2n} = \lim_{n\to\infty} \frac{1}{\left[\left(1+\frac{3}{n}\right)^n\right]^2} = \frac{1}{(e^3)^2} = e^{-6}.$$

1 (iii).

$$\lim_{n\to\infty} (n^2+1)\left(1-\cos\frac{1}{n}\right) = \lim_{n\to\infty} n^2\left(1-\cos\frac{1}{n}\right) + \lim_{n\to\infty} 1\cdot\left(1-\cos\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} n^2\left(1-\cos\frac{1}{n}\right) = \lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2},$$

where $x = \dfrac{1}{n}$. □

1 (iv).

$$\lim_{n\to\infty} (4^n+3^n)^{\frac{1}{2n}} = \lim_{n\to\infty} 4^{\frac{1}{2}}\left[1+\left(\frac{3}{4}\right)^n\right]^{\frac{1}{2n}}$$

$$= \lim_{n\to\infty} 2\left[1+\left(\frac{3}{4}\right)^n\right]^{\frac{1}{2n}} = 2\cdot(1+0)^0 = 2.$$

2 (i). Since $\dfrac{1}{\sqrt{n+1}}$ is positive, monotone deceasing and $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0$, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n+1}}$ is convergent by the Alternating Series Test. Since $\dfrac{1}{\sqrt{n+1}} \geq \dfrac{1}{2\sqrt{n}}$ and the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ is divergent by the p-series, the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{\sqrt{n+1}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$ is divergent by the Comparison Test. Thus the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n+1}}$ is conditionally convergent. □

2 (ii). Observe that

$$\left|(-1)^n\frac{3+\cos n}{n(\ln n)^{\frac{3}{2}}}\right| = \frac{|3+\cos n|}{n(\ln n)^{\frac{3}{2}}} \leq \frac{4}{n(\ln n)^{\frac{3}{2}}}.$$

Let $f(x) = \dfrac{4}{x(\ln x)^{\frac{3}{2}}}$. Then $f(x)$ is positive, continuous and monotone deceasing on $[2, +\infty)$. Since

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{4}{x(\ln x)^{\frac{3}{2}}}dx = \int_{\ln 2}^{\infty} \frac{4}{y^{\frac{3}{2}}}dy = \frac{4}{-\frac{3}{2}+1}y^{-\frac{3}{2}+1}\Big|_{\ln 2}^{\infty} = \frac{8}{\sqrt{\ln 2}},$$

where $y = \ln x$, is convergent, the series $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^{\frac{3}{2}}}$ is convergent the Integral Test. Thus the series $\displaystyle\sum_{n=2}^{\infty} \left|(-1)^n \frac{3+\cos n}{n(\ln n)^{\frac{3}{2}}}\right|$ is convergent by the Comparison Test and so the series $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{3+\cos n}{n(\ln n)^{\frac{3}{2}}}$ is absolutely convergent. □

2 (iii). Let $a_n = \dfrac{2^n (n!)^2}{(2n)!}$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1} \cdot [(n+1)!]^2 \cdot (2n)!}{(2n+2)! \cdot 2^n \cdot (n!)^2}$$

$$= \lim_{n\to\infty} \frac{2(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{2\left(1+1/n\right)^2}{(2+2/n)(2+1/n)} = \frac{2}{2\cdot 2} = \frac{1}{2} < 1.$$

Thus the positive series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (n!)^2}{(2n)!}$ is (absolutely) convergent by the ratio test. □

2 (iv). Since

$$\lim_{n\to\infty} \sqrt[n]{\left|(-2)^n \left(1 - \frac{1}{n}\right)^{n^2}\right|} = \lim_{n\to\infty} 2\left(1 - \frac{1}{n}\right)^n = 2e^{-1} < 1,$$

the series $\displaystyle\sum_{n=1}^{\infty} (-2)^n \left(1 - \frac{1}{n}\right)^{n^2}$ is absolutely convergent by the root test. □

3 (a). The radius of convergence

$$R = \frac{1}{\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\displaystyle\lim_{n\to\infty} \frac{n^2+n}{(n+1)^2+n+1}} = \frac{1}{\displaystyle\lim_{n\to\infty} \frac{1+1/n}{(1+1/n)^2+1/n+1/n^2}} = 1.$$

3 (b). The radius of convergence

$$R = \frac{1}{\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\displaystyle\lim_{n\to\infty} \frac{2^{n+1}\cdot(n+2)}{(n+3)\cdot 2^n}} = \frac{1}{\displaystyle\lim_{n\to\infty} \frac{2(1+2/n)}{1+3/n}} = \frac{1}{2}.$$

When $x = x_0 + R = 1 + \frac{1}{2}$, the series $\sum_{n=1}^{\infty} \frac{1}{n+2}$ is divergent because $\frac{1}{n+2} \geq \frac{1}{2n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series. When $x = x_0 - R = 1 - \frac{1}{2}$, the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+2}$ is convergent by the Alternating Series Test because $\frac{1}{n+2}$ is positive, monotone decreasing and $\lim_{n\to\infty} \frac{1}{n+2} = 0$. Thus the interval of convergence is $[1/2, 3/2)$. □

4 (a) (i). Observe

$$\frac{dy}{dx} = (x+2)e^y.$$

We have

$$\int e^{-y} dy = \int (x+2) dx + C.$$

Thus $-e^{-y} = x^2 + 2x + C$ or $y = -\ln(-x^2 - 2x - C)$. □

4 (a) (ii). From $r^2 + 2r + 3 = 0$, we have

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2} = -1 \pm \sqrt{2}i.$$

Thus

$$y(t) = Ae^{-t} \cos \sqrt{2}t + Be^{-t} \sin \sqrt{2}t.$$

□

4 (b).

$$\lim_{y\to 0} \frac{\arctan y - \sin y}{y^3 \cos y^2} = \lim_{y\to 0} \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)}{y^3 \cos y^2}$$
$$= \lim_{y\to 0} \frac{\left(-\frac{1}{3} + \frac{1}{3!}\right) + \left(\frac{1}{5} - \frac{1}{5!}\right) y^2 + \cdots}{\cos y^2} = \frac{-\frac{1}{3} + \frac{1}{3!}}{1} = -\frac{1}{6}.$$

□

5 (a). Let $F_n(x) = \left(\frac{x^3+1}{3}\right)^n \cos nx$. Since

$$0 \leq \left| \left(\frac{x^3+1}{3}\right)^n \cos nx \right| = \left| \left(\frac{x^3+1}{3}\right)^n \right| |\cos nx| \leq \left(\frac{2}{3}\right)^n \cdot 1 = \left(\frac{2}{3}\right)^n$$

for $0 \leq x \leq 1$,

$$-\left(\frac{2}{3}\right)^n \leq F_n(x) \leq \left(\frac{2}{3}\right)^n$$

for $0 \leq x \leq 1$. Since $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$,

$$F(x) = \lim_{n\to\infty} F_n(x) = 0$$

by the Squeeze Theorem. Observe that

$$0 \leq T_n = \sup_{0 \leq x \leq 1} |F_n(x) - F(x)| \leq \left(\frac{2}{3}\right)^n .$$

Since $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze Theorem and so the sequence of functions $\{F_n(x)\}$ converges uniformly to 0 on $[0,1]$ and so

$$\lim_{n\to\infty} \int_0^1 \left(\frac{x^3+1}{3}\right)^n \cos nx dx = \int_0^1 \lim_{n\to\infty} \left(\frac{x^3+1}{3}\right)^n \cos nx dx = \int_0^1 0 dx = 0.$$

□

5 (b). Let $f(x) = x^{\frac{1}{2}}$. Then

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f''(x) = -\frac{1}{2} \cdot \frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}}.$$

Thus

$$f(100) = \sqrt{100} = 10$$

$$f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

$$f''(100) = -\frac{1}{400\sqrt{100}} = -\frac{1}{4000}$$

and so the Taylor series is

$$f(100) + f'(100)(x-100) + \frac{f''(100)}{2!}(x-100)^2 + \cdots$$

$$= 10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2 + \cdots .$$

□

5(c). Given any $x_0 \in (0, \pi/2)$, let a and b be real numbers such that

$$0 < a < x_0 < b < \frac{\pi}{2}.$$

Then

1) Since

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{(k+1)^2 \sin^{k+1} x}{k^2 \sin^k x} = \lim_{k\to\infty} (1 + 1/k)^2 \sin x) = \sin x < 1$$

for $0 < x < \frac{\pi}{2}$, the series of functions $\sum_{k=1}^{\infty} k^2 \sin^k x$ converges pointwise on $[a,b] \subseteq (0, \pi/2)$.

2) Each term $k^2 \sin^k x$ is differentiable on $[a,b]$ and $\left(k^2 \sin^k x\right)' = k^3 \sin^{k-1} x \cos x$.

3) Observe that

$$\left|k^3 \sin^{k-1} x \cos x\right| = \left|k^3 \sin^{k-1} x\right| |\cos x| \leq k^3 \sin^{k-1} b \cdot 1 = k^3 \sin^{k-1} b$$

for $x \in [a, b]$. Since

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{(k+1)^3 \sin^k b}{k^3 \sin^{k-1} b} = \lim_{k\to\infty} (1 + 1/k)^3 \sin b) = \sin b < 1,$$

the series $\sum_{k=1}^{\infty} k^3 \sin^{k-1} b$ is convergent and so the series of functions

$$\sum_{k=1}^{\infty} k^3 \sin^{k-1} x \cos x$$

converges uniformly on $[a, b]$ by the Weierstrass M-test.

Thus the function $f(x) = \sum_{k=1}^{\infty} k^2 \sin^k x$ is differentiable on $[a, b]$ and so at x_0. Since x_0 is any point in $(0, \pi/2)$, $f(x)$ is differentiable on $(0, \pi/2)$ and

$$f'(x) = \sum_{k=1}^{\infty} k^3 \sin^{k-1} x \cos x$$

for $0 < x < \pi/2$. □

6 (a). Since

$$|a_k \cos kx| \leq |a_k|$$

and the series $\sum_{k=1}^{\infty} |a_k|$ is convergent, the series of functions $\sum_{k=1}^{\infty} a_k \cos kx$ converges uniformly on $[0, 2\pi]$ by the Weierstrass M-test. Thus

$$\int_0^{2\pi} \sum_{k=1}^{\infty} a_k \cos kx dx = \sum_{k=1}^{\infty} \int_0^{2\pi} a_k \cos kx dx = \sum_{k=1}^{\infty} \frac{a_k}{k} \sin kx \Big|_0^{2\pi} = \sum_{k=1}^{\infty} 0 = 0.$$

□

6 (b). From the equation $r^2 + 3r + 2 = 0$ or $(r + 1)(r + 2) = 0$, we have $r = -1, -2$ and so

$$y_h(t) = Ae^{-t} + Be^{-2t}.$$

Try $y_p(t) = tA_0 e^{-t}$. Then

$$y_p' = (A_0 - A_0 t)e^{-t}$$

$$y_p'' = -A_0 e^{-t} - (A_0 - A_0 t)e^{-t} = (-2A_0 + A_0 t)e^{-t}.$$

From the equation $y_p'' + 3y_p' + 2y_p = e^{-t}$, we have

$$(-2A_0 + A_0 t)e^{-t} + 3(A_0 - A_0 t)e^{-t} + 2tA_0 e^{-t} = e^{-t}$$

and so $A_0 = 1$. Hence $y_p = te^{-t}$ and the general solution

$$y(t) = y_h(t) = y_p(t) = Ae^{-t} + Be^{-2t} + te^{-t}.$$

Now $y'(t) = -Ae^{-t} - 2Be^{-2t} + e^{-t} - te^{-t}$. From $y(0) = 0$ and $y'(0) = 2$, we have the equations

$$\begin{cases} A + B + 0 = 0 \\ -A - 2B + 1 - 0 = 2 \end{cases}$$

and so $A = 1$ and $B = -1$. Thus the solution is

$$y(t) = e^{-t} - e^{-2t} + te^{-t}.$$

□

6 (c). Let $y(t)$ be the concentration of salt at time t. From time t to $t + \Delta t$, the salt in the tank roughly increases 0 and roughly decreases

$$y(t) \quad \text{lb/gal} \quad \times 10 \quad \text{gal/min} \quad \times \Delta t \quad \text{min}.$$

we have

$$100y(t + \Delta t) - 100y(t) \approx -y(t) \cdot 10 \cdot \Delta t$$
$$100\frac{y(t + \Delta t) - y(t)}{\Delta t} \approx -10y(t).$$

Let Δt tend to 0. We obtain the differential equation

$$100\frac{dy}{dt} = -10y \quad \text{or} \quad \frac{dy}{dt} = -\frac{1}{10}y.$$

and so the general solution is

$$y(t) = Ce^{-\frac{t}{10}}$$

Since $y(0) = 0.4$, we have $0.4 = C \cdot 1$ and so $C = 0.4$ or $y(t) = 0.4e^{-t/10}$. From

$$0.4e^{-t/10} \leq 0.1,$$

we have $-t/10 \leq \ln 1/4 = -\ln 4$ or $t \geq 10 \ln 4 \approx 13.86$. Thus about after 13.86 minutes the concentration will be reduced to 0.1. □

7 (a). Since

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

the Taylor series

$$\cos x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}.$$

□

7 (b).

$$\int_0^{\frac{1}{10}} \cos x^2 dx = \int_0^{\frac{1}{10}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} dx = \sum_{n=0}^{\infty} \int_0^{\frac{1}{10}} (-1)^n \frac{x^{4n}}{(2n)!} dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!(4n+1)} x^{4n+1} \Big|_0^{1/10} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+1) \cdot (2n)! \cdot 10^{4n+1}},$$

where

$$\int_0^{\frac{1}{10}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} dx = \sum_{n=0}^{\infty} \int_0^{\frac{1}{10}} (-1)^n \frac{x^{4n}}{(2n)!} dx$$

because the series of functions $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$ converges uniformly on $[0, 1/10]$. To see this, we have

$$\left| (-1)^n \frac{x^{4n}}{(2n)!} \right| \leq \frac{1}{(2n)! 10^{4n}}$$

for $0 \leq x \leq \frac{1}{10}$ and the series $\sum_{n=0}^{\infty} \frac{1}{(2n)! 10^{4n}}$ is convergent by the ratio test because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n)! 10^{4n}}{(2n+2)! 10^{4n+4}} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)10^4} = 0.$$

Thus the series of functions $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$ converges uniformly on $[0, 1/10]$ by the Weierstrass M-test. Now let

$$a_n = \frac{1}{(4n+1) \cdot (2n)! \cdot 10^{4n+1}}.$$

Then a_n is positive, monotone decreasing because

$$a_{n+1} = \frac{1}{(4n+5) \cdot (2n+2)! \cdot 10^{4n+5}} \leq a_n = \frac{1}{(4n+1) \cdot (2n)! \cdot 10^{4n+1}}$$

and $\lim_{n\to\infty} a_n = 0$ because

$$0 \leq a_n \leq \frac{1}{4n+1}$$

and $\lim_{n\to\infty} \frac{1}{4n+1} = 0$. From

$$a_{n+1} = \frac{1}{(4n+5) \cdot (2n+2)! \cdot 10^{4n+5}} \leq 10^{-6},$$

we have $n \geq 0$ and so

$$\int_0^{\frac{1}{10}} \cos x^2 dx \approx \frac{1}{10}$$

with error $\leq 10^{-6}$. □

7 (c). Let $S(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and $S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. Then

$$T_n = \sup_{-\infty<x<+\infty} |S_n(x) - S(x)| = \sup_{-\infty<x<+\infty} \left| \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right|$$

$$\geq \sup_{0<x<+\infty} \left| \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \right| = \sup_{0<x<+\infty} \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \geq \sup_{0<x<+\infty} \frac{x^{n+1}}{(n+1)!}.$$

Since

$$\lim_{x\to+\infty} \frac{x^{n+1}}{(n+1)!} = +\infty,$$

$$\sup_{0<x<+\infty} \frac{x^{n+1}}{(n+1)!} = +\infty$$

and so $T_n = +\infty$ for each $n \geq 0$. It follows that T_n does not tend to 0 and so the series of functions $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ does not converge uniformly on $(-\infty, +\infty)$.