2001/2002 Fall Term MA2108 Advanced Calculus II

## Answers to Optional Practice Questions for Chapter One

*Question 1 (a).*

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\sqrt{n+1}(\sqrt{n+8}-\sqrt{n})} &= \lim_{n\to\infty}\frac{\sqrt{n+8}+\sqrt{n}}{\sqrt{n+1}(\sqrt{n+8}-\sqrt{n})(\sqrt{n+8}+\sqrt{n})}\\ =\lim_{n\to\infty}\frac{1}{8}\left(\sqrt{\frac{n+8}{n+1}}+\sqrt{\frac{n}{n+1}}\right) &= \lim_{n\to\infty}\frac{1}{8}\left(\sqrt{\frac{1+8/n}{1+1/n}}+\sqrt{\frac{1}{1+1/n}}\right)\\ =\frac{1}{8}\left(\sqrt{\frac{1+0}{1+0}}+\sqrt{\frac{1}{1+0}}\right) &= \frac{1}{4}.\end{aligned}$$

$\square$

*Question 1 (b).*

$$\lim_{n\to\infty}\left(\frac{n}{n+3}\right)^{2n}=\lim_{n\to\infty}\frac{1}{\left(\frac{n+3}{n}\right)^{2n}}=\lim_{n\to\infty}\frac{1}{\left(\left(1+\frac{3}{n}\right)^{n}\right)^{2}}=\frac{1}{(e^{3})^{2}}=e^{-6}.$$

$\square$

*Question 1 (c).* Let $x=\dfrac{1}{n}$. Then

$$\begin{aligned}\lim_{n\to\infty}(n^2+1)\left(1-\cos\frac{1}{n}\right) &= \lim_{x\to 0}\left(\frac{1}{x^2}+1\right)(1-\cos x)=\lim_{x\to 0}\frac{(1+x^2)(1-\cos x)}{x^2}\\ &= \lim_{x\to 0}\frac{2x(1-\cos x)+(1+x^2)\sin x}{2x}\\ &= \lim_{x\to 0}\frac{2(1-\cos x)+2x\sin x+2x\sin x+(1+x^2)\cos x}{2}\\ &= \frac{2(1-1)+2\cdot 0\cdot 0+2\cdot 0\cdot 0+(1+0)\cdot 1}{2}=\frac{1}{2}.\end{aligned}$$

$\square$

*Question 2 (a).* First we show that $1\le a_n\le 2$ by induction on $n$. Since $a_1=1$, the statement holds for $n=1$. Suppose that $1\le a_{n-1}\le 2$ with $n\ge 2$. Then

$$\sqrt{1+1}\le\sqrt{1+a_{n-1}}=a_n\le\sqrt{1+2}$$

and so $1\le a_n\le 2$. The induction is finished and hence $\{a_n\}$ is bounded.

Now we show that $a_{n+1}-a_n\ge 0$ by induction on $n$. If $n=1$, we have $a_2-a_1=\sqrt{1+1}-1\ge 0$. Suppose that $a_n-a_{n-1}\ge 0$ with $n\ge 2$. Then

$$a_{n+1}-a_n=\sqrt{1+a_n}-\sqrt{1+a_{n-1}}=\frac{(\sqrt{1+a_n}-\sqrt{1+a_{n-1}})(\sqrt{1+a_n}+\sqrt{1+a_{n-1}})}{\sqrt{1+a_n}+\sqrt{1+a_{n-1}}}$$

$$= \frac{1+a_n-(1+a_{n-1})}{\sqrt{1+a_n}+\sqrt{1+a_{n-1}}} = \frac{a_n-a_{n-1}}{\sqrt{1+a_n}+\sqrt{1+a_{n-1}}} \geq 0.$$

The induction is finished. Thus $\{a_n\}$ is a monotone increasing bounded sequence and the limit of $\{a_n\}$ exits. Let $A=\lim_{n\to\infty} a_n$. Then we have

$$A=\lim_{n\to\infty} a_{n+1}=\lim_{n\to\infty}\sqrt{1+a_n}=\sqrt{1+A}$$

or $A^2=1+A$. Thus $A=\frac{1\pm\sqrt{5}}{2}$. Since $a_n\geq a_1=1$, we have $A=\lim_{n\to\infty} a_n\geq 1$ and so $A\neq\frac{1-\sqrt{5}}{2}$. Thus $\lim_{n\to\infty} a_n=A=\frac{1+\sqrt{5}}{2}$. □

*Question 2 (b).* First we show that $1\leq a_n\leq 2$ by induction on $n$. If $n=1$, we have $a_1=\sqrt{2}$ and so the statement holds for $n=1$. Suppose that $1\leq a_{n-1}\leq 2$ with $n\geq 2$. Then

$$\sqrt{2}\leq\sqrt{2a_{n-1}}=a_n\leq\sqrt{2\cdot 2}=2$$

and so $1\leq a_n\leq 2$. The induction is finished and so $\{a_n\}$ is a bounded sequence.

By using the fact $1\leq a_n\leq 2$ proved above, we have

$$a_n=\sqrt{2a_{n-1}}\geq\sqrt{a_{n-1}\cdot a_{n-1}}=a_{n-1}$$

for $n=2,3,4,\ldots$ and so the sequence $\{a_n\}$ is monotone increasing and bounded. Thus the limit of $a_n$ exists. Let $A=\lim_{n\to\infty} a_n$. Then we have

$$A=\lim_{n\to\infty} a_n=\lim_{n\to\infty}\sqrt{2a_{n-1}}=\sqrt{2A}$$

or $A^2=2A$. It follows that $A=0$ or $2$. Since $a_n\geq a_1=\sqrt{2}$ for all $n$, we have $A\geq\sqrt{2}>0$ and so $A\neq 0$. Thus $A=2$.

□

*Question 3.* Given any $\epsilon>0$, there exist positive integers $K_1$ and $K_2$ such that

$$|a_{2k-1}-A|<\epsilon$$

for all $k>K_1$ and

$$|a_{2k}-A|<\epsilon$$

for all $k>K_2$. Let $N=2\cdot\max\{K_1,K_2\}$. Let $n>N$. If $n=2k-1$ is odd, then $k>K_1$ because $2k-1>N\geq 2K_1$ and so $|a_{2k-1}-A|<\epsilon$. If $n=2k$ is even, then $k>K_2$ because $2k>N\geq 2K_2$ and so $|a_{2k}-A|<\epsilon$. In both cases, we have $|a_n-A|<\epsilon$ for $n>N$. Thus $\lim_{n\to\infty} a_n=A$. □

*Question 4.* Let $b_n=\sup\{a_n,a_{n+1},a_{n+2},\ldots\}$ and let $B_n=\sup\{a_n^2,a_{n+1}^2,a_{n+2}^2,\ldots\}$. Since $b_n\geq a_k$ for each $k\geq n$, we have $b_n^2\geq a_k^2$ for each $k\geq n$. Thus $b_n^2$ is an upper bound of $\{a_n^2,a_{n+1}^2,a_{n+2}^2,\ldots\}$ and so

$$b_n^2\geq B_n. \tag{1}$$

Since $B_n \geq a_k^2$ for each $k \geq n$, the nonnegative number $a_k \leq \sqrt{B_n}$ for each $k \geq n$. Thus $\sqrt{B_n}$ is an upper bound of $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$ and so

$$b_n \leq \sqrt{B_n}. \tag{2}$$

By Equations (1) and (2), we have $b_n^2 = B_n$. Thus

$$\overline{\lim_{n\to\infty}}\, a_n^2 = \lim_{n\to\infty} B_n = \lim_{n\to\infty} b_n^2 = \left(\lim_{n\to\infty} b_n\right)^2 = \left(\overline{\lim_{n\to\infty}}\, a_n\right)^2.$$

$\square$

*Question 5.* Suppose that $\lim\limits_{n\to\infty} |a_n| = 0$. Since $-|a_n| \leq a_n \leq |a_n|$, we have $\lim\limits_{n\to\infty} a_n = 0$ by the Squeeze theorem.

Conversely suppose that $\lim\limits_{n\to\infty} a_n = 0$. Given any $\epsilon > 0$, there exists a positive integer $N$ such that $|a_n| = |a_n - 0| < \epsilon$ for $n > N$ and so $||a_n| - 0| = |a_n| < \epsilon$ for $n > N$. Thus $\lim\limits_{n\to\infty} |a_n| = 0$. $\square$

*Question 6 (a).* Since $\inf A \cup B$ is a lower bound of $A \cup B$ and $A \subseteq A \cup B$, $\inf A \cup B$ is a lower bound of $A$ and so $\inf A \cup B \leq \inf A$. Similarly, $\inf A \cup B \leq \inf B$. It follows that $\inf A \cup B \leq \min\{\inf A, \inf B\}$.

Let $z \in A \cup B$. If $z \in A$, then $z \geq \inf A$. If $z \in B$, then $z \geq \inf B$. Thus $z \geq \min\{\inf A, \inf B\}$ for all $z \in A \cup B$ and so $\min\{\inf A, \inf B\}$ is a lower bound of $A \cup B$. Hence $\min\{\inf A, \inf B\} \leq \inf A \cup B$ and so $\min\{\inf A, \inf B\} = \inf A \cup B$. $\square$

*Question 6 (b).* No, it is not true in general. Counter example: Let $A = \{1, 2\}$ and let $B = \{0, 2\}$. Then $\inf A = \min A = 1$ and $\inf B = \min B = 0$ and so $\max\{\inf A, \inf B\} = 1$. On the other hand, $A \cap B = \{2\}$ and $\inf A \cap B = 2 \neq \max\{\inf A, \inf B\}$. $\square$