Optional Practice Questions for Chapter three

**1. a)**

$$S(x) = \lim_{n\to\infty} \frac{1}{nx+1} = \begin{cases} 1 & \text{if} \quad x = 0 \\ \lim\limits_{n\to\infty} \dfrac{1/n}{x+1/n} = 0 & \text{if} \quad 0 < x \le 1 \end{cases}$$

$\{S_n(x)\}$ does not converge uniformly to $S(x)$ on $[0,1]$ because each $S_n(x)$ is continuous on $[0,1]$ while its limiting function $S(x)$ is not.

**1. b)**

$$S(x) = \lim_{n\to\infty} x^n = 0$$

for $x \in (0,1)$. Observe that

$$T_n = \sup_{x\in(0,1)} |S_n(x) - S(x)| = \sup_{0<x<1} x^n = 1.$$

Since $\lim\limits_{n\to\infty} T_n = 1 \neq 0$, $\{S_n(x)\}$ does not converge uniformly to $S(x)$ on $(0,1)$.

**1. c)** $\{S_n(x)\}$ does not converge pointwise on $[0,1]$ because when $0 < x < 1$

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} n^2 x(1-x^n) = +\infty.$$

**1. d)**

$$S(x) = \lim_{n\to\infty} \frac{1-x^n}{1-x^n} = \begin{cases} \frac{1-0}{1+0} = 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x = 1 \\ \lim\limits_{n\to\infty} \dfrac{1/x^n - 1}{1/x^n + 1} = \dfrac{0-1}{0+1} = -1 & \text{if} \quad 1 < x < 2. \end{cases}$$

$S_n(x)$ does not converge uniformly to $S(x)$ because each $S_n(x)$ is continuous on $(0,2)$ while $S(x)$ is not.

**1. e)**

$$S(x) = \lim_{n\to\infty} \frac{x^{2n}}{1+x^{2n}} = \begin{cases} \frac{0}{1+0} = 0 & \text{if} \quad -1 < x < 1 \\ \frac{1}{1+1} = \frac{1}{2} & \text{if} \quad x = \pm 1 \\ \lim\limits_{n\to\infty} \dfrac{1}{1/x^{2n} + 1} = \dfrac{1}{0+1} = 1 & \text{if} \quad 1 < |x| < 2. \end{cases}$$

$S_n(x)$ does not converge uniformly to $S(x)$ because each $S_n(x)$ is continuous on $(0,2)$ while $S(x)$ is not.

**2. a)** By the geometric series, the series is convergent if and only if

$$\left|\frac{1-x}{1+x}\right| < 1 \quad \Leftrightarrow \quad -1 < \frac{1-x}{1+x} < 1.$$

From $-1 < \dfrac{1-x}{1+x}$, we have

$$0 < 1 + \frac{1-x}{1+x} = \frac{1+x+1-x}{1+x} = \frac{2}{1+x} \quad \Leftrightarrow \quad x > -1.$$

From $\dfrac{1-x}{1+x} < 1$, we have

$$0 > \frac{1-x}{1+x} - 1 = \frac{1-x-1-x}{1+x} = \frac{-2x}{1+x} \quad \Leftrightarrow \quad \frac{2x}{1+x} > 0$$
$$\Leftrightarrow x < -1 \quad \text{or} \quad x > 0.$$

Thus the domain is $x > 0$. Observe that

$$T_n = \sup_{x>0} |S_n(x) - S(x)| = \sup_{x>0} \left| \sum_{k=n+1}^{\infty} \left( \frac{1-x}{1+x} \right)^{k-1} \right|$$
$$= \sup_{x>0} \left| \left( \frac{1-x}{1+x} \right)^n + \left( \frac{1-x}{1+x} \right)^{n+1} + \left( \frac{1-x}{1+x} \right)^{n+2} + \cdots \right| = \sup_{x>0} \left| \frac{\left(\frac{1-x}{1+x}\right)^n}{1 - \frac{1-x}{1+x}} \right|$$
$$= \sup_{x>0} \frac{|1-x|^n}{2x|1+x|^{n-1}} = +\infty$$

because $\dfrac{|1-x|^n}{2x|1+x|^{n-1}} \to \infty$ as $x \to 0$. Since $T_n$ does not tend to 0, the series of functions $\displaystyle\sum_{k=1}^{\infty} \left( \frac{1-x}{1+x} \right)^{k-1}$ does NOT converge uniformly on $(0, +\infty)$.

2. b) This is a power series.

$$R = \frac{1}{\lim n \to \infty \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{n}{n+1}} = \frac{1}{\lim\limits_{n\to\infty} \frac{1}{1+1/n}} = 1.$$

Check the ending points. When $x = x_0 + R = 1$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the $p$-series. When $x = x_0 - R = -1$, the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by the alternating series test. Thus the domain is $[-1, 1)$.

Observe that

$$T_n = \sup_{-1 \le x < 1} |S_n(x) - S(x)| = \sup_{-1 \le x < 1} \left| \sum_{k=n+1}^{\infty} \frac{x^k}{k} \right| \ge \sup_{0<x<1} \left| \sum_{k=n+1}^{\infty} \frac{x^k}{k} \right|.$$

We have

$$T_{2^s} \ge \sup_{0<x<1} \left| \sum_{k=2^s+1}^{\infty} \frac{x^k}{k} \right| = \sup_{0<x<1} \sum_{k=2^s+1}^{\infty} \frac{x^k}{k} \ge \sup_{0<x<1} \sum_{k=2^s+1}^{2^{s+1}} \frac{x^k}{k}$$

$$= \sup_{0<x<1}\left(\frac{x^{2^s+1}}{2^s+1}+\frac{x^{2^s+2}}{2^s+2}+\cdots+\frac{x^{2^{s+1}}}{2^{s+1}}\right) \geq \sup_{0<x<1}\left(\frac{x^{2^s+1}}{2^{s+1}}+\frac{x^{2^s+2}}{2^{s+1}}+\cdots+\frac{x^{2^{s+1}}}{2^{s+1}}\right).$$

$$= \sup_{0<x<1}\frac{x^{2^s+1}}{2^{s+1}}\left(1+x+x^2+\cdots+x^{2^s-1}\right) \geq \frac{1}{2}$$

because

$$\lim_{x\to 1}\frac{x^{2^s+1}}{2^{s+1}}\left(1+x+x^2+\cdots+x^{2^s-1}\right) = \frac{1}{2^{s+1}}(1+1+\cdots+1) = \frac{1}{2^{s+1}}\cdot 2^s = \frac{1}{2}.$$

Thus the subsequence $\{T_{2^s}\}$ does not tend to 0 and so the sequence $\{T_n\}$ does not tend to 0. It follows that the series of functions $\sum_{k=1}^{\infty}\frac{x^k}{k}$ does NOT converge uniformly on $[-1, 1)$.

**2. c)** This is a power series.

$$R = \frac{1}{\lim n\to\infty \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim_{n\to\infty}\frac{n^2}{(n+1)^2}} = \frac{1}{\lim_{n\to\infty}\frac{1}{(1+1/n)^2}} = 1.$$

Check the ending points. When $x = x_0 + R = 1$, the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by the $p$-series. When $x = x_0 - R = -1$, the series $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ converges by the alternating series test. Thus the domain is $[-1, 1]$. Since

$$\left|\frac{x^k}{k^2}\right| \leq \frac{1}{k^2}$$

for $x \in [-1, 1]$ and the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by the $p$-series, the series of functions $\sum_{k=1}^{\infty}\frac{x^k}{k^2}$ converges uniformly on $[-1, 1]$ by the $M$-test.

**2. d)** Since

$$\left|\frac{\cos kx}{k^2}\right| \leq \frac{1}{k^2}$$

and the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by the $p$-series, the series of functions $\sum_{k=1}^{\infty}\frac{\cos kx}{k^2}$ converges uniformly on $(-\infty, +\infty)$ by the $M$-test and the domain is $(-\infty, +\infty)$.

**2. e)** Observe that

$$\sum_{k=1}^{\infty}(\cos k\pi)x^{2k} = \sum_{k=1}^{\infty}(-1)^k x^{2k} = \sum_{k=1}^{\infty}(-x^2)^k.$$

By the geometric series, the domain is $|-x^2| < 1$ or $-1 < x < 1$.

$$T_n = \sup_{-1<x<1} |S_n(x) - S(x)| = \sup_{-1<x<1} \left| \sum_{k=n+1}^{\infty} (-x^2)^k \right| = \sup_{-1<x<1} \left| \frac{(-x^2)^{n+1}}{1-(-x^2)} \right|$$

$$= \sup_{-1<x<1} \frac{x^{2(n+1)}}{1+x^2} \geq \frac{1}{1+1} = \frac{1}{2}$$

because $\lim_{x\to 1} \frac{x^{2(n+1)}}{1+x^2} = \frac{1}{2}$. Thus $T_n$ does not tend to 0 and so the series of functions $\sum_{k=1}^{\infty} (\cos k\pi) x^{2k}$ does not converge uniformly on $(-1, 1)$.

**3.** Let $R_n = \sum_{k=n}^{\infty} a_k$ for each $n$. Since $\sum_{k=1}^{\infty} a_k$ converges, the sequence $R_n = S - S_{n-1} \to 0$ as $n \to \infty$. For any $\epsilon > 0$, there exists $N$ such that $|R_n| < \frac{\epsilon}{3}$ for $n > N$.

Observe that $a_k = (a_k + a_{k+1} + \cdots) - (a_{k+1} + a_{k+2} + \cdots) = R_k - R_{k+1}$ and

$$\sum_{k=n}^{n+q} a_k x^k = \sum_{k=n}^{n+q} (R_k - R_{k+1}) x^k$$

$$= (R_n - R_{n+1})x^n + (R_{n+1} - R_{n+2})x^{n+1} + \cdots + (R_{n+q} - R_{n+q+1})x^{n+q}$$

$$= R_n x^n + R_{n+1}(-x^n + x^{n+1}) + R_{n+2}(-x^{n+1} + x^{n+2}) + \cdots + R_{n+q}(-x^{n+q-1} + x^{n+q}) - R_{n+q+1}x^{n+q}$$

$$= R_n x^n + x^n(-1+x) \cdot \left(R_{n+1} + R_{n+2}x + R_{n+3}x^2 + \cdots + R_{n+q}x^{q-1}\right) - R_{n+q+1}x^{n+q}$$

$$= R_n x^n - R_{n+q+1}x^{n+q} + x^n(-1+x) \sum_{k=0}^{q-1} R_{n+1+k} x^k.$$

When $n > N$, we have

$$\left| \sum_{k=n}^{n+q} a_k x^k \right| = \left| R_n x^n - R_{n+q+1}x^{n+q} + x^n(-1+x) \sum_{k=0}^{q-1} R_{n+1+k} x^k \right|$$

$$\leq |R_n| x^n + |R_{n+q+1}| x^{n+q} + x^n |-1+x| \sum_{k=0}^{q-1} |R_{n+1+k}| x^k$$

$$< \frac{\epsilon}{3} x^n + \frac{\epsilon}{3} x^{n+q} + x^n(1-x) \sum_{k=0}^{q-1} \frac{\epsilon}{3} x^k$$

$$= \frac{\epsilon}{3} x^n + \frac{\epsilon}{3} x^{n+q} + \frac{\epsilon}{3} x^n(1-x^q) = \frac{2\epsilon}{3} x^n \leq \frac{2\epsilon}{3}$$

for any $q$ and $x \in [0, 1]$. Let $q$ tends to $\infty$. We obtain

$$\left| \sum_{k=n}^{\infty} a_k x^k \right| \leq \frac{2\epsilon}{3}$$

for $n > N$ and $x \in [0,1]$ and so

$$T_n = \sup_{0\le x\le 1} |S_n(x) - S(x)| = \sup_{0\le x\le 1} \left| \sum_{k=n+1}^{\infty} a_k x^k \right| \le \frac{2\epsilon}{3} < \epsilon$$

for $n > N$. It follows that $\lim_{n\to\infty} T_n = 0$ and so the series of functions $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[0,1]$.

**4.** Since the series $\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}$, the series of functions $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$ converges uniformly on $[0,1]$. Let

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}.$$

Then $f(x)$ is continuous on $[0,1]$. Observe that

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{k=0}^{\infty} (-1)^k t^{2k} dt = \sum_{k=0}^{\infty} (-1)^k \int_0^x t^{2k} dt = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

for $-1 < x < 1$, where $\int_0^x \sum_{k=0}^{\infty} = \sum_{k=0}^{\infty} \int_0^x$ by Theorem 3.7.3. Thus

$$f(x) = \arctan x$$

for $0 \le x < 1$ and so

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = f(1) = \lim_{x\to 1} f(x) = \lim_{x\to 1} \arctan x = \arctan 1 = \frac{\pi}{4}.$$

**5.** Let $f(x) = \sin x$. By the Taylor formula

$$E = \frac{f^{(6)}(\xi)}{6!} x^6$$

where $\xi$ is between $x$ and 0. Since $f^{(6)}(x) = -\sin x$, we have

$$|E| = \left| \frac{f^{(6)}(\xi)}{6!} x^6 \right| = \frac{\sin(\xi) x^6}{6!} \le \frac{\xi \cdot x^6}{6!} \le \frac{\frac{1}{2}\left(\frac{1}{2}\right)^6}{6!} = \frac{1}{2^7 \cdot 6!} = \frac{1}{128 \cdot 720}$$

$$= \frac{1}{92160} < 1.0850695 \times 10^{-5}$$

for $x \in [0, 0.5]$.

**6.** From the geometric series $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, we have

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} x^{2k}$$

for $|x| < 1$. By taking the derivatives, we have

$$\left(\frac{1}{1-x^2}\right)' = \left(\sum_{k=0}^{\infty} x^{2k}\right)' = \sum_{k=1}^{\infty} 2k \cdot x^{2k-1},$$

where $\left(\sum_{k=0}^{\infty} x^{2k}\right)' = \sum_{k=1}^{\infty} 2k \cdot x^{2k-1}$ because

i) $\left(x^{2k}\right)' = 2kx^{2k-1}$ exists and continuous.

ii) the series of functions $\sum_{k=0}^{\infty} x^{2k}$ converges pointwise on $(-1, 1)$.

iii) by theorem 3.7.3, the series $\sum_{k=1}^{\infty} 2k \cdot x^{2k-1}$ converges uniformly on $[-\rho, +\rho]$ for any $\rho < 1$.

Observe that

$$\left(\frac{1}{1-x^2}\right)' = -\frac{1}{\left(1-x^2\right)^2} \cdot (-2x) = \frac{2x}{\left(1-x^2\right)^2}.$$

Thus

$$\frac{x}{\left(1-x^2\right)^2} = \sum_{k=1}^{\infty} k \cdot x^{2k-1}$$

for $|x| < 1$. In particular

$$\sum_{k=1}^{\infty} \frac{k}{2^{2k-1}} = \frac{\frac{1}{2}}{\left(1-\frac{1}{4}\right)^2} = \frac{8}{9}.$$