

1. i) Given any $x \in (-1, 1)$. If $x = 0$, clearly $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$. For $x \neq 0$, we may assume that $0 < x < 1$. Let $f_n(t) = (-1)^n t^n$. Since $|f_n(t)| \leq x^n$ for $0 \leq t \leq x$ and $\sum_{n=0}^{\infty} x^n$ is convergent by the geometric series, the series of functions $\sum_{n=0}^{\infty} f_n(t)$ converges uniformly on $[0, x]$ and so

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{1}{1+t} dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}. \end{aligned}$$

1. ii)

$$\ln(1+2x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2x^2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2^n}{n} x^{2n}$$

for $|x| < \frac{1}{2}$.

2. i)

$$\begin{aligned} \int_0^{0.2} \sin x^2 dx &= \int_0^{\frac{1}{5}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x^2)^{2n-1}}{(2n-1)!} dx = \sum_{n=1}^{\infty} \int_0^{\frac{1}{5}} (-1)^{n+1} \frac{x^{4n-2}}{(2n-1)!} dx \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(\frac{1}{5}\right)^{4n-1}}{(2n-1)! \cdot 4n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)! \cdot (4n-1) \cdot 5^{4n-1}}, \end{aligned}$$

where $\int_0^{\frac{1}{5}} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \int_0^{\frac{1}{5}}$ because the series of functions $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n-2}}{(2n-1)!}$ converges uniformly on $\left[0, \frac{1}{5}\right]$. Let $a_n = \frac{1}{(2n-1)! \cdot (4n-1) \cdot 5^{4n-1}}$. Then a_n is positive, monotone decreasing and $\lim_{n \rightarrow \infty} a_n = 0$. By applying the alternating series estimation, from

$$a_{n+1} = \frac{1}{(2n+1)! \cdot (4n+3) \cdot 5^{4n+3}} < 10^{-8},$$

we have $n \geq 2$. Thus

$$\int_0^{0.2} \sin x^2 dx \approx \frac{1}{3 \cdot 5^3} - \frac{1}{3! \cdot 7 \cdot 5^7} = \frac{1}{375} - \frac{1}{3281250} \approx 0.002666362$$

with error $< 10^{-8}$.

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2. ii)

$$\begin{aligned} \int_0^{0.1} \frac{1}{\sqrt{(1+x^4)}} dx &= \int_0^{\frac{1}{10}} (1+x^4)^{-\frac{1}{2}} dx = \int_0^{\frac{1}{10}} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k} dx \\ &= \sum_{k=0}^{\infty} \int_0^{\frac{1}{10}} \binom{-\frac{1}{2}}{k} x^{4k} dx = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{1}{(4k+1) \cdot 10^{4k+1}}, \end{aligned}$$

where $\int_0^{\frac{1}{5}} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \int_0^{\frac{1}{5}}$ because the series of functions $\sum_{n=1}^{\infty} \binom{-\frac{1}{2}}{k} x^{4k}$ converges uniformly on $\left[0, \frac{1}{10}\right]$ by Theorem 3.7.3. Observe that

$$\binom{-\frac{1}{2}}{k} = (-1)^k \frac{(2k-1)!!}{2^k \cdot k!}.$$

The series

$$\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{1}{(4k+1) \cdot 10^{4k+1}} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k-1)!!}{2^k \cdot k! \cdot (4k+1) 10^{4k+1}}$$

is an alternating series. From

$$\frac{(2k+1)!!}{2^{k+1} \cdot (k+1)! \cdot (4k+5) \cdot 10^{4k+5}} < 10^{-8},$$

we have $k \geq 1$ and so

$$\int_0^{0.1} \frac{1}{\sqrt{(1+x^4)}} dx \approx \frac{1}{10} - \frac{1}{2 \cdot 5 \cdot 10^5} = \frac{1}{10} - \frac{1}{10^6} = 0.099999$$

with error $< 10^{-8}$.

3. i) Recall that

$$\arctan y = y - \frac{y^3}{3} + \frac{y^5}{5} - \dots,$$

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots,$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots.$$

$$\begin{aligned} \lim_{y \rightarrow \infty} \frac{\arctan y - \sin y}{y^3 \cos y} &= \lim_{y \rightarrow \infty} \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \dots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right)}{y^3 \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right)} \\ &= \lim_{y \rightarrow \infty} \frac{\left(-\frac{1}{3} + \frac{1}{3!}\right) + \left(\frac{1}{5} - \frac{1}{5!}\right) y^2 + \dots}{1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots} = -\frac{1}{3} + \frac{1}{3!} = -\frac{1}{6}. \end{aligned}$$

3. ii) Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \dots.$$

$$\lim_{x \rightarrow \infty} x^2(e^{-1/x^2} - 1) = \lim_{x \rightarrow \infty} x^2 \left(1 - \frac{1}{x^2} + \frac{1}{2!x^4} - \dots - 1 \right) = \lim_{x \rightarrow \infty} -1 + \frac{1}{2!x^2} - \dots = -1.$$

4. i) Recall that

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1}.$$

From

$$a_{k+1} = \frac{1}{2k+1} < 10^{-3},$$

we have $k \geq 500$ and so we need at least 500 terms.

4. ii) For $|x| < 1$,

$$\begin{aligned} \arcsin x &= \int_0^x \frac{1}{\sqrt{1-t^2}} dt = \int_0^x (1+(-t^2))^{-\frac{1}{2}} dt = \int_0^x \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-t^2)^k dt \\ &= \sum_{k=0}^{\infty} \int_0^x \binom{-\frac{1}{2}}{k} (-1)^k t^{2k} dt = \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} \frac{x^{2k+1}}{2k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k (-1)^k \frac{(2k-1)!!}{2^k \cdot k! \cdot (2k+1)} x^{2k+1}. \end{aligned}$$

Thus

$$\frac{\pi}{6} = \arcsin \frac{1}{2} = \sum_{k=0}^{\infty} \frac{(2k-1)!!}{2^k \cdot k! \cdot (2k+1) \cdot 2^{2k+1}}.$$

The remainder

$$\begin{aligned} R_n = |S - S_n| &= \sum_{k=n+1}^{\infty} \frac{(2k-1)!!}{2^k \cdot k! \cdot (2k+1) \cdot 2^{2k+1}} < \sum_{k=n+1}^{\infty} \frac{1}{(2k+1)2^{2k+1}} \\ &< \sum_{k=n+1}^{\infty} \frac{1}{(2n+3)2^{2k+1}} = \frac{1}{(2n+3)2^{2n+3}} \left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots \right) \\ &= \frac{1}{(2n+3)2^{2n+3} \left(1 - \frac{1}{4} \right)} = \frac{1}{3(2n+3)2^{2n+1}} \end{aligned}$$

Let

$$\frac{1}{3(2n+3)2^{2n+1}} < 10^{-3} \Leftrightarrow 3(2n+3)2^{2n+1} > 1000$$

We obtain $n \geq 3$ and so

$$\frac{\pi}{6} \approx \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{3}{2^2 \cdot 2 \cdot 5 \cdot 2^5} + \frac{5 \cdot 3}{2^3 \cdot 3! \cdot 7 \cdot 2^7} = \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \frac{5}{7 \times 2^{11}}.$$

with error less than 0.001.

5. From

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(x) = \sum_{k=0}^{\infty} \binom{\frac{1}{5}}{k} x^{3k},$$

we have $\frac{f^{(30)}(0)}{30!} = \binom{\frac{1}{5}}{10}$ or

$$f^{(30)}(0) = 30! \binom{\frac{1}{5}}{10} = \frac{30! \cdot \frac{1}{5} \cdot (\frac{1}{5} - 1) \cdots (\frac{1}{5} - 9)}{10!}.$$

6.

$$\begin{aligned} \sqrt{101} &= (100 + 1)^{\frac{1}{2}} = 10 \left(1 + \frac{1}{100}\right)^{\frac{1}{2}} = 10 \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \frac{1}{100^k} \\ &= 10 + \sum_{k=1}^{\infty} \frac{\frac{1}{2} \cdot (\frac{1}{2} - 1) \cdots (\frac{1}{2} - k + 1)}{k! \cdot 10^{2k-1}} \\ &= 10 + \frac{1}{20} + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{(2k-3)!!}{k! 2^k \cdot 10^{2k-1}} \end{aligned}$$

Let $a_k = \frac{(2k-3)!!}{k! \cdot 2^k \cdot 10^{2k-1}}$ for $k \geq 2$. Then $a_k > 0$. Since

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{(2k-1)!! \cdot k! 2^k \cdot 10^{2k-1}}{(k+1)! \cdot 2^{k+1} \cdot 10^{2k+1} \cdot (2k-3)!!} \\ &= \frac{2k-1}{(k+1) \cdot 2 \cdot 10^2} \leq 1, \end{aligned}$$

$a_{k+1} \leq a_k$ or $\{a_k\}$ is monotone decreasing. Observe that

$$0 \leq a_k = \frac{\frac{1}{2} \cdot (1 - \frac{1}{2}) \cdots (k-1 - \frac{1}{2})}{k! \cdot 10^{2k-1}} \leq \frac{1 \cdot 2 \cdots k}{k! \cdot 10^{2k-1}} = \frac{1}{10^{2k-1}}$$

for $k \geq 2$. Since $\lim_{k \rightarrow \infty} \frac{1}{10^{2k-1}} = 0$, $\lim_{k \rightarrow \infty} a_k = 0$ by the Squeeze theorem. Thus we can apply the alternating series estimation. From

$$a_{k+1} = \frac{(2k-1)!!}{(k+1)! \cdot 2^{k+1} \cdot 10^{2k+1}} < 10^{-6},$$

we have $k \geq 2$. Thus

$$\sqrt{101} \approx 10 + \frac{1}{20} - \frac{1}{8000} = 1.049875$$

with an error of magnitude less than 10^{-6} .