

2 a).

$$y = Ce^{\int -\sin t dt} = Ce^{\cos t}$$

with $C \in \mathbb{R}$. □

2 b). From $(1+t^2)\frac{dy}{dt} + ty = 0$, we have

$$\begin{aligned} y_h &= Ce^{-\int \frac{t}{1+t^2} dt} = Ce^{-\frac{1}{2} \int \frac{1}{1+t^2} d(1+t^2)} = Ce^{-\frac{1}{2} \ln(1+t^2)} \\ &= C \left(e^{\ln(1+t^2)} \right)^{-\frac{1}{2}} = C(1+t^2)^{-\frac{1}{2}} = \frac{C}{\sqrt{1+t^2}} \end{aligned}$$

with $C \in \mathbb{R}$. Let

$$y_p = \frac{C(t)}{\sqrt{1+t^2}},$$

where $C(t)$ is an undetermined function. Then

$$y_p' = \frac{C'(t)\sqrt{1+t^2} - C(t) \cdot \frac{2t}{2\sqrt{1+t^2}}}{1+t^2}$$

and so we have the equation

$$(1+t^2) \frac{C'(t)\sqrt{1+t^2} - C(t) \cdot \frac{2t}{2\sqrt{1+t^2}}}{1+t^2} + t \frac{C(t)}{\sqrt{1+t^2}} = (1+t^2)^{3/2}.$$

It follows that

$$C'(t)\sqrt{1+t^2} = (1+t^2)^{3/2} \quad \text{or} \quad C'(t) = 1+t^2.$$

Thus $C(t) = t + \frac{t^3}{3}$, and so

$$y(t) = y_h(t) + y_p(t) = \frac{C}{\sqrt{1+t^2}} + \frac{t + \frac{t^3}{3}}{\sqrt{1+t^2}} = \frac{C + t + \frac{t^3}{3}}{\sqrt{1+t^2}}$$

with $C \in \mathbb{R}$. □

3 a). The integrating factor

$$\mu(t) = e^{\int g(t) dt} = e^{\int 4t dt} = e^{2t^2}.$$

Thus the general solution

$$\begin{aligned} y &= \frac{1}{\mu(t)} \left(\int \mu(t)f(t) dt + C \right) = \frac{1}{e^{2t^2}} \left(\int 2te^{2t^2} \cdot e^{-2t^2} dt + C \right) \\ &= \frac{1}{e^{2t^2}} \left(\int 2t dt + C \right) = e^{-2t^2} (t^2 + C). \end{aligned}$$

From $y(1) = 5$, we have

$$5 = e^{-2}(1+C) \quad \text{or} \quad C = 5e^2 - 1$$

and so $y(t) = e^{-2t^2} (t^2 + 5e^2 - 1)$. □

3 b). From $(1+t^2)\frac{dy}{dt} - 2ty^4 = 0$, we have

$$\int \frac{dy}{y^4} = \int \frac{2t}{1+t^2} dt + C$$

$$-\frac{y^{-3}}{3} = \ln(1+t^2) + C.$$

Since $y(0) = \frac{1}{2}$, $\frac{(\frac{1}{2})^{-3}}{3} = \ln(1+0) + C$ or

$$C = -\frac{8}{3}.$$

Thus $\frac{y^{-3}}{-3} = \ln(1+t^2) - \frac{8}{3}$ or

$$y^3 = \frac{1}{8 - 3\ln(1+t^2)}$$

and so $y = \frac{1}{\sqrt[3]{8 - 3\ln(1+t^2)}}$ □

3 c) From $\sqrt{1+t^3}\frac{dy}{dt} - 6t^2e^{-y} = 0$, we have

$$\int e^y dy = \int \frac{6t^2}{\sqrt{1+t^3}} dt + C = 2 \int \frac{d(1+t^3)}{\sqrt{1+t^3}} + C = 2 \cdot 2\sqrt{1+t^3} + C.$$

Thus

$$e^y = 4\sqrt{1+t^3} + C.$$

Since $y(0) = 0$, $e^0 = 4\sqrt{1+0} + C$, and so $C = -3$. Hence $e^y = 4\sqrt{1+t^3} - 3$ or $y = \ln(4\sqrt{1+t^3} - 3)$. □

4 a). Let $v = \frac{y}{t}$. Then $y = vt$ and so $y' = v't + v$. From the equation $y' = \frac{y}{t} + \frac{y^2}{t^2}$, we have

$$tv' + v = v + v^2 \quad \text{or} \quad tv' = v^2.$$

Thus

$$\int \frac{dv}{v^2} = \int \frac{dt}{t} + C$$

$$-\frac{1}{v} = \ln(|t|) + C \quad \text{or} \quad -\frac{t}{y} = \ln(|t|) + C$$

and so $y = -\frac{t}{\ln(|t|) + C}$ □

4 b) Let $v = \frac{y}{t}$. Then $y = vt$ and so $y' = v't + v$. From the equation $y' = \frac{y}{t} - \tan \frac{y}{t}$, we have

$$tv' + v = v - \tan v.$$

Thus

$$\begin{aligned}\int \frac{\cos v dv}{\sin v} &= -\int \frac{dt}{t} + k \\ \int \frac{d \sin v}{\sin v} &= -\ln |t| + k \\ \ln |\sin v| &= -\ln |t| + k \\ |\sin v| &= e^{-\ln |t| + k} = e^k \cdot \frac{1}{|t|} \\ \sin v &= \frac{C}{t}\end{aligned}$$

and so $\sin \frac{y}{t} = \frac{C}{t}$ or $y = t \arcsin \left(\frac{C}{t} \right)$ □

5 a). From

$$r^2 - 7r + 10 = 0,$$

we have $r = 2, 5$ and so $y = Ae^{2t} + Be^{5t}$ □

5 b). From

$$r^2 + 6r + 9 = 0,$$

we have $(r + 3)^2 = 0$ or $r_1 = r_2 = -3$ and so $y = Ae^{-3t} + Bte^{-3t}$ □

5 c). From

$$r^2 + 2r + 3 = 0,$$

we have

$$r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 3}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{2}i.$$

Thus $y = Ae^{-t} \cos \sqrt{2}t + Be^{-t} \sin \sqrt{2}t$. □

6. From

$$r^2 + 2r + 1 = 0,$$

we have $(r + 1)^2 = 0$ or $r = r_1 = r_2 = -1$. Thus $y = Ae^{-t} + Bte^{-t}$ and

$$y' = -Ae^{-t} - Bte^{-t} + Be^{-t}.$$

Since $y(0) = 1$ and $y'(0) = 0$, we have

$$\begin{cases} Ae^0 + B \cdot 0 \cdot e^0 = 1 \\ -Ae^0 - B \cdot 0 \cdot e^0 + Be^0 = 0. \end{cases}$$

and so $A = 1$ and $B = A = 1$. The solution is $y = e^{-t} + te^{-t}$. □

7 a). Since $c = 3 \neq 0$, try

$$y_p = At^3 + Bt^2 + Ct + D.$$

Then $y'_p = 3At^2 + 2Bt + C$ and $y''_p = 6At + 2B$. From the equation $y''_p + 3y_p = 6t^3 - 6$, we have

$$6At + 2B + 3(At^3 + Bt^2 + Ct + D) = 6t^3 - 6$$

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and so

$$\begin{cases} 3A = 6 \\ 3B = 0 \\ 6A + 3C = 0 \\ 2B + 3D = -6. \end{cases}$$

Thus $A = 2$, $B = 0$, $C = -4$ and $D = -2$, and so $y_p = 2t^3 - 4t - 2$ \square

7 b). From $r^2 + 3 = 0$, we have $r = \pm\sqrt{-3} = \pm\sqrt{3}i$. Try

$$y_p = t \cdot [A \cos(\sqrt{3}t) + B \sin(\sqrt{3}t)].$$

Then

$$y'_p = A \cos(\sqrt{3}t) + B \sin(\sqrt{3}t) + t[-A \sin(\sqrt{3}t)\sqrt{3} + B \cos(\sqrt{3}t)\sqrt{3}]$$

$$\begin{aligned} y''_p &= 2[-A \sin(\sqrt{3}t)\sqrt{3} + B \cos(\sqrt{3}t)\sqrt{3}] + t[-A \cos(\sqrt{3}t)(\sqrt{3})^2 - B \sin(\sqrt{3}t)(\sqrt{3})^2] \\ &= 2[-A \sin(\sqrt{3}t)\sqrt{3} + B \cos(\sqrt{3}t)\sqrt{3}] - 3t[A \cos(\sqrt{3}t) + B \sin(\sqrt{3}t)] \end{aligned}$$

Since $y''_p + 3y_p = \cos(\sqrt{3}t)$, we have

$$\begin{aligned} &2[-A \sin(\sqrt{3}t)\sqrt{3} + B \cos(\sqrt{3}t)\sqrt{3}] - 3t[A \cos(\sqrt{3}t) + B \sin(\sqrt{3}t)] \\ &+ 3t \cdot [A \cos(\sqrt{3}t) + B \sin(\sqrt{3}t)] = \cos(\sqrt{3}t) \end{aligned}$$

and so

$$\begin{cases} -2A\sqrt{3} = 0 \\ 2B\sqrt{3} = 1 \end{cases}$$

It follows that $y_p = \frac{1}{2\sqrt{3}}t \sin(\sqrt{3}t)$ \square

8 a). From $r^2 - 3r + 2 = 0$, we have $(r - 2)(r - 1) = 0$ and so $r = 1, 2$. It follows that

$$y_h = Ae^t + Be^{2t}.$$

Try

$$y_p = t \cdot A_0e^{2t} + B_0e^{3t}.$$

We have

$$\begin{aligned} y'_p &= 2A_0te^{2t} + A_0e^{2t} + 3B_0e^{3t} \\ y''_p &= 2A_0e^{2t} + 4A_0te^{2t} + 2A_0e^{2t} + 9B_0e^{3t} \\ &= 4A_0te^{2t} + 4A_0e^{2t} + 9B_0e^{3t}. \end{aligned}$$

Since $y''_p - 3y'_p + 2y_p = 3e^{2t} + e^{3t}$, we have

$$\begin{aligned} 4A_0te^{2t} + 4A_0e^{2t} + 9B_0e^{3t} - 3(2A_0te^{2t} + A_0e^{2t} + 3B_0e^{3t}) + 2(A_0te^{2t} + B_0e^{3t}) &= 3e^{2t} + e^{3t} \\ (4A_0 - 3A_0)e^{2t} + (9B_0 - 9B_0 + 2B_0)e^{3t} &= 3e^{2t} + e^{3t} \end{aligned}$$

Thus $A_0 = 3$ and $B_0 = \frac{1}{2}$, and so $y(t) = y_h + y_p = Ae^t + Be^{2t} + 3te^{2t} + \frac{1}{2}e^{3t}$ \square

8 b). From $r^2 + 4 = 0$, $r = \pm\sqrt{-4} = \pm 2i$. Thus $y_h = A \cos 2t + B \sin 2t$. Try

$$y_p = t(A_0 \cos 2t + B_0 \sin 2t) + C_0 \sin t.$$

We have

$$\begin{aligned} y_p' &= A_0 \cos 2t + B_0 \sin 2t + t(-2A_0 \sin 2t + 2B_0 \cos 2t) + C_0 \cos t \\ y_p'' &= -2A_0 \sin 2t + 2B_0 \cos 2t + (-2A_0 \sin 2t + 2B_0 \cos 2t) \\ &\quad + t(-4A_0 \cos 2t - 4B_0 \sin 2t) - C_0 \sin t \\ &= -4A_0 \sin 2t + 4B_0 \cos 2t - 4t(A_0 \cos 2t + B_0 \sin 2t) - C_0 \sin t. \end{aligned}$$

Since $y_p'' + 4y_p = \cos 2t + \sin t$,

$$\begin{aligned} &-4A_0 \sin 2t + 4B_0 \cos 2t - 4t(A_0 \cos 2t + B_0 \sin 2t) - C_0 \sin t \\ &\quad + 4[t(A_0 \cos 2t + B_0 \sin 2t) + C_0 \sin t] = \cos 2t + \sin t \\ &-4A_0 \sin 2t + 4B_0 \cos 2t + 3C_0 \sin t = \cos 2t + \sin t. \end{aligned}$$

Thus $A_0 = 0$, $B_0 = \frac{1}{4}$ and $C_0 = \frac{1}{3}$, and so

$$y = y_h + y_p = A \cos 2t + B \sin 2t + \frac{1}{4}t \sin 2t + \frac{1}{3} \sin t \quad \square$$

8 c). From $r^2 - 2r + 1 = 0$, we have $(r + 1)^2 = 0$ or $r = r_1 = r_2 = 1$. Thus $y_1 = e^t$, $y_2 = te^t$ and

$$y_h = Ae^t + Bte^t.$$

Note that $a = 1$ and $g(t) = \frac{e^t}{t}$.

$$\begin{aligned} y_p(t) &= \left(\int \frac{gy_2}{ay_1'y_2 - ay_1y_2'} dt \right) \cdot y_1 + \left(\int \frac{-y_1g}{ay_1'y_2 - ay_1y_2'} dt \right) \cdot y_2 \\ &= \left(\int \frac{\frac{e^t}{t}te^t}{e^t te^t - e^t(te^t + e^t)} dt \right) \cdot e^t + \left(\int \frac{-e^t \cdot \frac{e^t}{t}}{e^t te^t - e^t(te^t + e^t)} dt \right) \cdot te^t \\ &= \int (-1)dt \cdot e^t + \int \frac{1}{t}dt \cdot te^t = -te^t + te^t \ln |t|. \end{aligned}$$

Thus

$$y(t) = Ae^t + Bte^t - te^t + te^t \ln |t| = Ae^t + B'te^t + te^t \ln |t|,$$

where $B' = B - 1$ is any constant and so is A . □

9. From $r^2 + 4r + 3 = 0$, we have $(r + 3)(r + 1) = 0$ or $r = -1, -3$. Thus

$$y_h = Ae^{-t} + Be^{-3t}.$$

Try

$$y_p = t \cdot A_0 e^{-3t}.$$

We have

$$\begin{aligned} y_p' &= A_0 e^{-3t} + A_0 t e^{-3t} \cdot (-3) = (A_0 - 3A_0 t) e^{-3t} \\ y_p'' &= -3A_0 e^{-3t} + (A_0 - 3A_0 t) e^{-3t} \cdot (-3) = (-6A_0 + 9A_0 t) e^{-3t}. \end{aligned}$$

Since $y_p'' + 4y_p' + 3y_p = 2e^{-3t}$,

$$\begin{aligned} &(-6A_0 + 9A_0 t) e^{-3t} + 4(A_0 - 3A_0 t) e^{-3t} + 3A_0 t e^{-3t} = 2e^{-3t} \\ &-2A_0 e^{-3t} = 2e^{-3t}. \end{aligned}$$

Thus $A_0 = -1$ and $y(t) = y_h + y_p = Ae^{-t} + Be^{-3t} - te^{-3t}$, and

$$y' = -Ae^{-t} - 3Be^{-3t} - e^{-3t} + 3te^{-3t}.$$

Since $y(0) = 0$ and $y'(0) = 3$,

$$\begin{cases} Ae^0 + Be^0 - 0 = 0 \\ -Ae^0 - 3Be^0 - e^0 + 0 = 3. \end{cases}$$

Thus $B = -2$ and $A = 2$, and so $y(t) = 2e^{-t} - 3e^{-3t} - te^{-3t}$ □

10. First we set up a differential equation. Let $y(t)$ denote the concentration of salt in the tank at time t . At time t we have

$$200 \cdot y(t) \text{ lb}$$

of salt. From time t to $t + \Delta t$, the salt in the tank roughly increases

$$\frac{1}{2} \text{ lb} \times 4 \text{ gal/min} \times \Delta t \text{ min}$$

and roughly decreases

$$y(t) \text{ lb/gal} \times 4 \text{ gal/min} \times \Delta t \text{ min}.$$

Thus we have

$$200y(t + \Delta t) - 200y(t) \approx \frac{1}{2} \cdot 4 \cdot \Delta t - y(t) \cdot 4 \cdot \Delta t$$

$$200 \frac{y(t + \Delta t) - y(t)}{\Delta t} \approx \frac{1}{2} \cdot 4 - y(t) \cdot 4.$$

Let Δt tend to 0. We obtain the differential equation

$$200 \frac{dy}{dt} = 2 - 4y \quad \text{or} \quad \frac{dy}{dt} + \frac{1}{50}y = \frac{1}{100}.$$

Now we solve this differential equation. From $\frac{dy}{dt} + \frac{1}{50}y = 0$, we have

$$y_h = Ce^{-\int \frac{1}{50} dt} = Ce^{-\frac{1}{50}t}.$$

Observe that $y_p = \frac{1}{2}$ is a solution of the nonhomogeneous equation. Thus the general solution is

$$y(t) = y_h + y_p = Ce^{-\frac{1}{50}t} + \frac{1}{2}.$$

Since $y(0) = \frac{S_0}{200}$,

$$\frac{S_0}{200} = C \cdot e^0 + \frac{1}{2}.$$

Thus $C = \frac{S_0}{200} - \frac{1}{2}$ and so $y(t) = \frac{S_0}{200}e^{-\frac{1}{50}t} + \frac{1}{2} \left(1 - e^{-\frac{1}{50}t}\right)$ □