Solutions to Tutorial 3

Question 1 (i). Let $a_n = \ln \frac{n+2}{n+3}$. Then the partial sum

$$\begin{aligned} S_n = a_1 + a_2 + \cdots + a_n &= \ln\frac{1+2}{1+3} + \ln\frac{2+2}{2+3} + \ln\frac{3+2}{3+3} + \cdots + \ln\frac{n+2}{n+3} \\ &= \ln\frac{(1+2)(2+2)(3+2)(4+2)\cdots(n-1+2)(n+2)}{(1+3)(2+3)(3+3)(4+3)\cdots(n-1+3)(n+3)} \\ &= \ln\frac{3\cdot 4\cdot 5\cdot 6\cdots(n+1)\cdot(n+2)}{4\cdot 5\cdot 6\cdot 7\cdots(n+2)\cdot(n+3)} = \ln\frac{3}{n+3} = \ln 3 - \ln(n+3). \end{aligned}$$

Thus $\{S_n\}$ is divergent and so is the series $\sum_{n=1}^{\infty} \ln\frac{n+2}{n+3}$. □

Question 1 (ii). Let $a_n = \frac{1}{n(n+2)}$. Observe that

$$\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

The partial sum

$$\begin{aligned} S_n &= \frac{1}{1\cdot(1+2)} + \frac{1}{2\cdot(2+2)} + \cdots + \frac{1}{n\cdot(n+2)} \\ &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n} - \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n+2}\right)\right] \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right). \end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n\to\infty} S_n = \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4}.$$

□

Question 2 (a). Let $a_n = \frac{n^2-1}{2n^2+n}$. Then $\lim_{n\to\infty} a_n = \frac{1}{2} \neq 0$ and so the series $\sum_{n=1}^{\infty} \frac{n^2-1}{2n^2+n}$ is divergent by the divergence test. □

Question 2 (b). Let $a_n = \sin \frac{n\pi}{2}$. Then $\lim_{n\to\infty} a_n$ does not exist and so the series $\sum_{n=1}^{\infty} \sin \frac{n\pi}{2}$ is divergent by the divergence test. $\square$

Question 2 (c). Let $a_n = \frac{n^2 + 1 + \ln n}{n + n^3 + 4}$ and let $b_n = \frac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n + n^3 + 4}{n^2 + 1 + \ln n} = \lim_{n\to\infty} \frac{\frac{1}{n^2} + 1 + \frac{4}{n^3}}{1 + \frac{1}{n^2} + \frac{\ln n}{n^2}} = \frac{0 + 1 + 0}{1 + 0 + 0} = 1.$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, so is $\sum_{n=1}^{\infty} \frac{n^2 + 1 + \ln n}{n + n^3 + 4}$ by the limit comparison test. $\square$

Question 2 (d). Observe that

$$\frac{3 + \sin n}{n^2} \leq \frac{4}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{4}{n^2} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p-series, the positive series $\sum_{n=1}^{\infty} \frac{3 + \sin n}{n^2}$ is convergent by the comparison test. $\square$

Question 2 (e). Observe that

$$\frac{2^n + 3}{3^{n+1} - n} \leq \frac{2^n + 2^n}{3^{n+1}} = \frac{2^{n+1}}{3^{n+1}} = \left(\frac{2}{3}\right)^{n+1}$$

for $n \geq 2$. Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+1}$ is convergent by the geometric series, the positive series $\sum_{n=1}^{\infty} \frac{2^n + 3}{3^{n+1} - n}$ is convergent by the comparison test. $\square$

Question 2 (f). Let $a_n = \frac{2}{n^{1+\frac{1}{n}}}$ and let $b_n = \frac{1}{n}$. Observe that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1}{n} \cdot \frac{n^{1+\frac{1}{n}}}{2} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\sum_{n=1}^{\infty} \frac{2}{n^{1+\frac{1}{n}}}$ is divergent by the limit comparison test. $\square$

Question 2 (g). Observe that

$$\frac{4 + (-1)^n}{2n} \geq \frac{3}{2n}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\displaystyle\sum_{n=1}^{\infty} \frac{4+(-1)^n}{2n}$ is divergent by the comparison test. $\square$

Question 2 (h). Observe that

$$\frac{1}{n(1+\ln n)^p} = \frac{(1+\ln n)^{-p}}{n} \geq \frac{1}{n}$$

for $p \leq 0$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)^p}$ is divergent for $p \leq 0$ by the comparison test. $\square$

Question 2 (i). Observe that

$$\frac{n}{n^2+1} \geq \frac{n}{n^2+n^2} = \frac{n}{2n^2} = \frac{1}{2n}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the positive series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent by the comparison test. $\square$