

Question 1 (b). Let $a_n = \frac{1}{n^2}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} \sqrt{a_n}$ is divergent by the p -series. \square

Question 2. Let $f(x) = \frac{1}{x(1 + \ln x)}$. Then $f(x)$ is a positive, continuous monotone decreasing function over $[1, +\infty)$. Observe that

$$\int f(x)dx = \int \frac{1}{x(1 + \ln x)}dx = \int \frac{1}{1 + y}dy = \ln(1 + y) + C = \ln(1 + \ln x) + C,$$

where $y = \ln x$. The integral $\int_1^{\infty} f(x)dx$ is divergent and so is the series $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)}$ by the integral test. \square

Question 3 (b). Let $a_n = \frac{1}{n^x + 1}$. If $x > 0$, then a_n is positive, monotone decreasing and $\lim_{n \rightarrow \infty} a_n = 0$. Thus the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x + 1}$ is convergent by the alternating series test. If $x = 0$, then $a_n = \frac{1}{2}$ and so $\lim_{n \rightarrow \infty} a_n = \frac{1}{2}$. If $x < 0$, then $\lim_{n \rightarrow \infty} a_n = 1$. Thus the limit $\lim_{n \rightarrow \infty} (-1)^{n+1} a_n$ does not exist for $x \leq 0$ and so the series $g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x + 1}$ is divergent by the divergence test. In summary, the domain of the function $g(x)$ is $(0, +\infty)$. \square

Question 4 (a). Let $a_n = \frac{(3n)!}{6^n n! (2n)!}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{[3(n+1)]! 6^n n! (2n)!}{6^{n+1} (n+1)! [2(n+1)]! (3n)!} \\ &= \lim_{n \rightarrow \infty} \frac{(3n+3)(3n+2)(3n+1)}{6(n+1)(2n+2)(2n+1)} = \frac{3 \cdot 3 \cdot 3}{6 \cdot 2 \cdot 2} = \frac{27}{24} > 1. \end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \frac{(3n)!}{6^n n! (2n)!}$ is divergent by the ratio test. \square

Question 4 (b). Let $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} 2 \left(1 - \frac{1}{n}\right)^n = \frac{2}{e} < 1$, the series $\sum_{n=1}^{\infty} a_n$ is convergent by the ratio test. \square

Question 5 (a). Let $a_n = \frac{5n^2 \cdot 3^n}{4^{n+4}}$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{5^{\frac{1}{n}} (\sqrt[n]{n})^2 \cdot 3}{4 \cdot 4^{\frac{4}{n}}} = \frac{1 \cdot 1^2 \cdot 3}{4 \cdot 1} = \frac{3}{4} < 1.$$

Thus the series $\sum_{n=1}^{\infty} \frac{5n^2 \cdot 3^n}{4^{n+4}}$ is convergent by the simplified root test. \square

Question 5 (b). Let $a_n = \frac{3^{2n}}{5^n} \left(1 - \frac{1}{2n}\right)^{n^2}$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{3^2}{5} \left(1 + \frac{-\frac{1}{2}}{n}\right)^n = \frac{9}{5} e^{-\frac{1}{2}} = \frac{9}{5\sqrt{e}} > 1$$

because $e < \frac{9^2}{5^2} = \frac{81}{25} = 3.24$. Thus the series $\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n} \left(1 - \frac{1}{2n}\right)^{n^2}$ is divergent by the simplified root test. \square

Question 5 (c). Let a_n be the n -term in the series. Then $a_{2n-1} = \frac{1}{4^{2n-1}}$ and $a_{2n} = \frac{1}{5^{2n}}$. Thus

$$\sqrt[n]{a_n} = \begin{cases} \frac{1}{4} & \text{if } n \text{ is odd} \\ \frac{1}{5} & \text{if } n \text{ is even} \end{cases}$$

and so $\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{4} < 1$. Hence the series is convergent by the root test. \square

Question 6 (a). Let $a_n = \sqrt{2n+2} - \sqrt{n}$. Then

$$a_n = \frac{(\sqrt{2n+2} - \sqrt{n})(\sqrt{2n+2} + \sqrt{n})}{\sqrt{2n+2} + \sqrt{n}} = \frac{2n+2-n}{\sqrt{2n+2} + \sqrt{n}} = \frac{n+2}{\sqrt{2n+2} + \sqrt{n}} \rightarrow \infty$$

as $n \rightarrow \infty$. The series $\sum_{n=1}^{\infty} \sqrt{2n+2} - \sqrt{n}$ is divergent by the divergence test. \square

Question 6 (b). Let $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \cdot \frac{2^n}{5^n}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1) \cdot 2^{n+1} \cdot n! \cdot 5^n}{(n+1)! \cdot 5^{n+1} \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot 2^n} \\ &= \lim_{n \rightarrow \infty} \frac{(2n+1) \cdot 2}{(n+1) \cdot 5} = \frac{2 \cdot 2}{1 \cdot 5} = \frac{4}{5} < 1. \end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \cdot \frac{2^n}{5^n}$ is convergent by the ratio test. \square

Question 6 (c). Let $a_n = \frac{\ln n}{n^{1.2}}$ and let $b_n = \frac{1}{n^{1.1}}$. Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\ln n \cdot n^{1.1}}{n^{1.2}} = \lim_{n \rightarrow \infty} \frac{\ln n}{n^{0.1}} = 0.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ is convergent by the p -series, the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^{1.2}}$ is convergent by the limit comparison test for the case $a_n \ll b_n$. \square

Question 6 (d). Let $a_n = \left(\frac{n}{n+2}\right)^{n^2}$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+2}\right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{2}{n}\right)^n} = \frac{1}{e^2} < 1.$$

Thus the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2}$ is convergent by the simplified root test. \square

Question 6 (e). Let $a_n = \frac{1}{n}$ and let $b_n = \frac{1}{(\ln n)^3}$. Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(\ln n)^3}{n} = 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by the harmonic series, the series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$ is divergent by the limit comparison test for the case $a_n \ll b_n$. \square

Question 6 (f). Let $a_n = \left(\frac{4}{9} + \frac{n^3}{3^n}\right)^{\frac{n}{2}}$. Then

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \left(\frac{4}{9} + \frac{n^3}{3^n}\right)^{\frac{1}{2}} = \left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{2}{3} < 1$$

and so the series is convergent by the simplified root test. \square