Solutions to Tutorial 5

Question 1. Let $a_n = (-1)^n \frac{\cos n}{2^n}$. Then $|a_n| \leq \frac{1}{2^n}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent by the geometric series, the series $\sum_{n=1}^{\infty} \left|(-1)^n \frac{\cos n}{2^n}\right|$ is convergent by the comparison test and so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{2^n}$ is absolutely convergent. □

Question 2 (i). Let $a_n = \frac{\ln n}{\sqrt{n}}$. Then $a_n \geq 0$. We show that a_n is eventually monotone decreasing. Let $f(x) = \frac{\ln x}{\sqrt{x}}$. Then

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln x \frac{1}{2\sqrt{x}}}{x} = \frac{2 - \ln x}{2x^{\frac{3}{2}}} \leq 0$$

for $x \geq e^2$ and so $\{a_n\}$ is monotone decreasing for $n \geq 9$. Since $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}}$ is convergent by the alternating series test. □

Question 2 (ii). Since $\left|(-1)^{n+1} \frac{\ln n}{\sqrt{n}}\right| \geq \frac{1}{n^{\frac{1}{2}}}$ for $n \geq 3$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ is divergent by the p-series, the series $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{\ln n}{\sqrt{n}}\right|$ is divergent by the comparison test. By (i), the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{\sqrt{n}}$ is conditionally convergent. □

Question 3 (a). This series is conditionally convergent because it is convergent by the alternating series test and the series $\sum_{n=1}^{\infty} \left|(-1)^n \frac{3}{2n+1}\right|$ is divergent by the limit comparison test with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. □

Question 3 (b). Let $a_n = (-1)^n \frac{n}{4n+3}$. Then $\lim_{n\to\infty} a_{2n-1} = -\frac{1}{4}$ and $\lim_{n\to\infty} a_{2n} = \frac{1}{4}$. Thus the limit of $(-1)^n \frac{n}{4n+3}$ does not exist and so the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{4n+3}$ is divergent by the divergence test. □

Question 3 (c). Let $a_n = (-1)^n \left(\frac{1+2n}{3+4n}\right)^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1+2n}{3+4n} = \frac{2}{4} = \frac{1}{2} < 1.$$

Thus the positive series $\sum_{n=1}^{\infty} |a_n|$ is convergent by the simplified root test and so the series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1+2n}{3+4n}\right)^n$ is absolutely convergent. □

Question 3 (d). This series is conditionally convergent because it is convergent by the alternating series test but the series $\sum_{n=2}^{\infty} \left|(-1)^{n+1}\frac{1}{n\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{n\ln n}$ is divergent by the integral test. □

Question 4. We use integral test estimation for solving this question. Let $f(x) = \frac{1}{x^5}$. From $\int_n^{\infty} f(x)dx = \frac{1}{-4}x^{-4}\Big|_n^{+\infty} = \frac{1}{4n^4} < 0.001$, we have $n > \sqrt[4]{\frac{1000}{4}} = 3.976$ or $n \geq 4$. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.036$$

with the error less than 0.001. □

Question 5. We use alternating series test estimation for solving this question. Let $a_n = \frac{1}{n^5}$. From $a_{n+1} = \frac{1}{(n+1)^5} < 0.001$, we have $n+1 > \sqrt[5]{1000}$ or $n \geq 3$ and so

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n^5} \approx 1 - \frac{1}{2^5} + \frac{1}{3^5} \approx 0.9729$$

with error less than 0.001. □

Question 6 (a). We use the ratio test for general series for solving this question. Let $a_n = \frac{x^n}{n^{\frac{3}{2}}}$. From

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}}|x|} = |x| < 1,$$

we have $-1 < x < 1$. We check the ending points $x = \pm 1$. When $x = -1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$ is convergent by the alternating series test. When $x = 1$, then the series

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is convergent by the p-series. Thus the domain of the function $f(x)$ is the closed internal $[-1, 1]$. □

Question 6 (b). We use the ratio test for general series for solving this question. Let $a_n = \frac{(-1)^n (x-1)^n}{2n+1}$. From

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-1|^{n+1}(2n+1)}{(2n+3)|x-1|^n} = |x-1| < 1,$$

we have $0 < x < 2$. We check the ending points $x = 0$ or 2. When $x = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{2n+1} = \sum_{n=1}^{\infty} \frac{1}{2n+1}$ is divergent by the limit comparison test with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. When $x = 2$, then the series $\sum_{n=1}^{\infty} \frac{(-1)^n(2-1)^n}{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ is convergent by the alternating series test. Thus the domain of the function $g(x)$ is the internal $(0, 2]$. □