

1 (i). Let $F_n(x) = \frac{n + e^x}{n + x^2}$. Then the limiting function

$$F(x) = \lim_{n \rightarrow \infty} F_n(x) = \lim_{n \rightarrow \infty} \frac{n + e^x}{n + x^2} = \lim_{n \rightarrow \infty} \frac{1 + \frac{e^x}{n}}{1 + \frac{x^2}{n}} = 1$$

and

$$\begin{aligned} 0 \leq T_n &= \sup_{0 \leq x \leq 1} |F_n(x) - F(x)| = \sup_{0 \leq x \leq 1} \left| \frac{n + e^x}{n + x^2} - 1 \right| \\ &= \sup_{0 \leq x \leq 1} \frac{|e^x - x^2|}{n^2 + x^2} \leq \sup_{0 \leq x \leq 1} \frac{e^x + x^2}{n^2 + x^2} \leq \frac{e + 1}{n^2} \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{e + 1}{n^2} = \lim_{n \rightarrow \infty} 0 = 0$, $\lim_{n \rightarrow \infty} T_n = 0$ by the Squeeze Theorem. Thus $\{F_n\}$ converges uniformly to $F(x)$ and so

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{n + e^x}{n + x^2} dx = \int_0^1 \lim_{n \rightarrow \infty} F_n(x) dx = \int_0^1 1 dx = 1.$$

1 (ii). Let $F_n(x) = \left(\frac{x^2 + 1}{8}\right)^n \sin nx$. Then

$$0 \leq |F_n(x)| = \left| \left(\frac{x^2 + 1}{8}\right)^n \sin nx \right| \leq \left(\frac{x^2 + 1}{8}\right)^n = \left(\frac{5}{8}\right)^n$$

for $1 \leq x \leq 2$. Since $\lim_{n \rightarrow \infty} \left(\frac{5}{8}\right)^n = \lim_{n \rightarrow \infty} 0 = 0$, the limiting function $F(x) = 0$ for $1 \leq x \leq 2$ and

$$\lim_{n \rightarrow \infty} T_n = \lim_{n \rightarrow \infty} \sup_{1 \leq x \leq 2} |F_n(x) - F(x)| = \lim_{n \rightarrow \infty} \sup_{1 \leq x \leq 2} |F_n(x)| = 0$$

Thus $\{F_n\}$ converges uniformly to $F(x)$ and so

$$\lim_{n \rightarrow \infty} \int_1^2 \left(\frac{x^2 + 1}{8}\right)^n \sin nx dx = \int_1^2 \lim_{n \rightarrow \infty} F_n(x) dx = \int_1^2 0 dx = 0.$$

2 (i). Since

$$\left| \frac{\cos nx}{n^2 + x^2} \right| \leq \frac{1}{n^2}$$

and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the p -series, the series of functions $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + x^2}$

converges uniformly on $(-\infty, +\infty)$ by the Weierstrass M -test.

2 (ii). Since

$$\left| \frac{1}{1 + n^3 x^2} \right| \leq \frac{1}{1 + 2^2 n^3} \leq \frac{1}{n^3}$$

for $x \geq 2$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the p -series, the series of functions

$\sum_{n=1}^{\infty} \frac{1}{1+n^3x^2}$ converges uniformly on $[2, +\infty)$ by the Weierstrass M -test.

2 (iii). Let $f_n(x) = \frac{xe^{-x}}{n^2}$ for $x \in (0, +\infty)$. Then

$$f'_n(x) = \frac{1}{n^2} (e^{-nx} - xne^{-nx}) = \frac{(1-nx)e^{-nx}}{n^2}.$$

Thus $f_n(x)$ is monotone increasing for $0 \leq x \leq \frac{1}{n}$ and monotone decreasing for $x \geq \frac{1}{n}$. It follows that

$$|f_n(x)| \leq f_n\left(\frac{1}{n}\right) = \frac{\frac{1}{n}e^{-n \cdot \frac{1}{n}}}{n^2} = \frac{e^{-1}}{n^3}$$

for $x \in (0, +\infty)$. Since the series $\sum_{n=1}^{\infty} \frac{e^{-1}}{n^3} = e^{-1} \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the p -series,

the series of functions $\sum_{n=1}^{\infty} \frac{xe^{-nx}}{n^2}$ converges uniformly on $(0, +\infty)$ by the Weierstrass M -test.

3. Let $f_k(x) = \frac{(-1)^k x^k}{1+x^{2k}}$ for $0 < x < \frac{2}{3}$. Since

$$|f_k(x)| = \left| \frac{(-1)^k x^k}{1+x^{2k}} \right| \leq \frac{\left(\frac{2}{3}\right)^k}{1} = \left(\frac{2}{3}\right)^k$$

for $0 < x < \frac{2}{3}$ and the series $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ converges by the geometric series, the series

of functions $\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{1+x^{2k}}$ converges uniformly on $\left(0, \frac{2}{3}\right)$ by the Weierstrass M -test.

Thus the function $F(x) = \sum_{n=1}^{\infty} \frac{(-1)^k x^k}{1+x^{2k}}$ is continuous on the interval $\left(0, \frac{2}{3}\right)$.