1. Let $f_n(x) = \dfrac{x^n(1-x^2)}{\sqrt{1+x}}$ for $0 \le x \le \dfrac{1}{2}$. Since

$$|f_n(x)| = \frac{|x|^n|1-x^2|}{\sqrt{1+x}} \le \frac{\left(\frac{1}{2}\right)^n \cdot 1}{1} = \left(\frac{1}{2}\right)^n$$

for $0 \le x \le \dfrac{1}{2}$ and the series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ is convergent by the geometric series, the series of functions $\displaystyle\sum_{n=0}^{\infty}\frac{x^n(1-x^2)}{\sqrt{1+x}}$ converges uniformly on $[0, \frac{1}{2}]$ by the M-test. Note that each $f_n(x)$ is Riemann integrable on $[0, \frac{1}{2}]$. We have

$$\sum_{n=0}^{\infty}\int_0^{\frac{1}{2}}\frac{x^n(1-x^2)}{\sqrt{1+x}}dx = \int_0^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{x^n(1-x^2)}{\sqrt{1+x}}dx = \int_0^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}x^n\right)\cdot\frac{1-x^2}{\sqrt{1+x}}dx$$

$$= \int_0^{\frac{1}{2}}\frac{1}{1-x}\cdot\frac{1-x^2}{\sqrt{1+x}}dx = \int_0^{\frac{1}{2}}\sqrt{1+x}dx = \frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^{\frac{1}{2}} = \frac{2}{3}\left[\left(\frac{3}{2}\right)^{\frac{3}{2}} - 1\right] = \sqrt{\frac{3}{2}} - \frac{2}{3}.$$

2(i). Since

$$|a_k \sin kx| \le |a_k|$$

for $x \in (-\infty, +\infty)$ and $\displaystyle\sum_{k=1}^{\infty}|a_k|$ is convergent, the series of functions $\displaystyle\sum_{k=1}^{\infty}a_k\sin kx$ converges uniformly on $(-\infty, +\infty)$ by the M-test.

2 (ii). By (i), the series of functions $\displaystyle\sum_{k=1}^{\infty}a_k\sin kx$ converges uniformly on $[0, 2\pi]$. Since each $a_k \sin kx$ is Riemann integrable, we have

$$\int_0^{2\pi}\sum_{k=1}^{\infty}a_k\sin kx dx = \sum_{k=1}^{\infty}\int_0^{2\pi}a_k\sin kx dx = \sum_{k=1}^{\infty}0 = 0.$$

3. From $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for any x, we have

$$e^{-x^3} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{n!}.$$

Since

$$\left|\frac{(-1)^n x^{3n}}{n!}\right| \le \frac{1}{n!}$$

for $0 \le x \le 1$ and the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent by the ratio test, the series of functions $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$ converges uniformly to e^{-x^3} on $[0,1]$ by the M-test. Note that each $\frac{(-1)^n x^{3n}}{n!}$ is Riemann integrable. Thus

$$\int_0^1 e^{-x^3} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{3n}}{n!} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(3n+1)}.$$

Let $a_n = \frac{1}{n!(3n+1)}$. Then the sequence $\{a_n\}$ is positive, monotone decreasing and $\lim_{n\to\infty} a_n = 0$. By applying the alternating test estimation, from $a_{n+1} < 0.001$ or $(n+1)!(3n+4) \ge 1000$, we have $n \ge 4$ and so

$$\int_0^1 e^{-x^3} dx \approx 1 - \frac{1}{1!\cdot 4} + \frac{1}{2!\cdot 7} - \frac{1}{3!\cdot 10} + \frac{1}{4!\cdot 13} = 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312}$$

with error less than 0.001.

4. Let $f_n(x) = \frac{\cos^n x}{n^3}$. Given any point x_0 in $(-\infty, +\infty)$, let a and b be any numbers such that $a < x_0 < b$. Then

(1). Each $f_n'(x) = \frac{-\cos^{n-1} x \sin x}{n^2}$ is continuous on $[a,b]$.

(2). The series of functions $\sum_{n=1}^{\infty} f_n(x)$ absolutely converges on $[a,b]$ by the comparison test because $\left|\frac{\cos^n x}{n^3}\right| \le \frac{1}{n^3}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent by the p-series. So it converges pointwise on $[a,b]$.

(3). The series of functions $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[a,b]$ by the M-test because $\left|\frac{-\cos^{n-1} x \sin x}{n^2}\right| \le \frac{1}{n^2}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series.

Thus the function $f(x) = \sum_{n=1}^{\infty} \frac{\cos^n x}{n^3}$ is differentiable on $[a,b]$ and so at x_0. Since x_0 is any given point in $(-\infty, +\infty)$, the function $f(x)$ is differentiable on $(-\infty, +\infty)$.

5. Let $f_n(x) = \frac{x^{2n}}{(2n)!}$. Given any point $x_0 \in (-\infty, +\infty)$, let a and b be numbers such that $a < x_0 < b$. Note that

(1). $f_0'(x) = 0$ and $f_n'(x) = \frac{x^{2n-1}}{(2n-1)!}$ for $n \ge 1$ are continuous on $[a,b]$.

(2). The series of functions $\sum_{n=0}^{\infty} f_n(x)$ absolutely converges on $[a, b]$ by the ratio test. So it converges pointwise on $[a, b]$.

(3). The series of functions $\sum_{n=0}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$ converges uniformly on $[a, b]$ by the M-test because

$$\left| \frac{x^{2n-1}}{(2n-1)!} \right| \le \frac{(\max\{|a|, |b|\})^{2n-1}}{(2n-1)!}$$

and the series $\sum_{n=1}^{\infty} \frac{(\max\{|a|, |b|\})^{2n-1}}{(2n-1)!}$ converges by the ratio test.

Thus the function $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ is differentiable on $[a, b]$ and so at x_0. Thus $f(x)$ is differentiable on $(-\infty, +\infty)$ with

$$f'(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

for $x \in (-\infty, +\infty)$.

Similarly, the function $f'(x)$ is differentiable with

$$f''(x) = \sum_{n=1}^{\infty} \left(\frac{x^{2n-1}}{(2n-1)!} \right)' = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(2n-2)!} = f(x)$$

and so $y = f(x)$ is a solution to $y'' = y$.

6. Let $S(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. Then, for $x > 0$, we have

$$|S_n(x) - S(x)| = \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \ge \frac{x^{n+1}}{(n+1)!}$$

because each $\frac{x^k}{k!} \ge 0$. Thus

$$T_n = \sup_{-\infty < x < \infty} |S_n(x) - S(x)| \ge \sup_{0 < x < +\infty} |S_n(x) - S(x)| \ge \sup_{0 < x < +\infty} \frac{x^{n+1}}{(n+1)!} = +\infty$$

because $x^{n+1} \to +\infty$ as $x \to \infty$. Since T_n does not tend to 0, the sequence of the functions $S_n(x)$ does not converge uniformly to $S(x)$ on $(-\infty, +\infty)$ and so the series of functions $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ does not converge uniformly on $(-\infty, +\infty)$.