1 i).

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\limsup \sqrt[n]{\left(1+\frac{3}{n}\right)^{n^2}}} = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{3}{n}\right)^{n}} = \frac{1}{e^3}.$$

1 ii)

$$R = \frac{1}{\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty}\frac{|-3|^{n+1}\cdot n!}{(n+1)!\cdot|-3|^n}} = \frac{1}{\lim\limits_{n\to\infty}\frac{3}{n+1}} = +\infty.$$

1 iii) Since $a_n = \left(\frac{1}{5}\right)^n$ if n is odd and $\left(\frac{1}{6}\right)^n$ if n is even, $\sqrt[n]{|a_n|} = \frac{1}{5}$ if n is odd and $\frac{1}{6}$ if n is even. Thus

$$b_n = \sup_{k\geq n}\sqrt[n]{|a_n|} = \frac{1}{5}$$

and so $\limsup \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} b_n = \frac{1}{5}$. It follows that

$$R = \frac{1}{\limsup\sqrt[n]{|a_n|}} = 5.$$

1 iv) Observe that

$$\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n^2} = \sum_{n=1}^{\infty}\frac{3^n}{n^2}\cdot(x-\frac{2}{3})^n.$$

Thus

$$R = \frac{1}{\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty}\frac{3^{n+1}\cdot n^2}{(n+1)^2\cdot 3^n}} = \frac{1}{\lim\limits_{n\to\infty}\frac{3}{\left(1+\frac{1}{n}\right)^2}} = \frac{1}{3}.$$

2 i).

$$R = \frac{1}{\lim\limits_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty}\frac{|-2|^{n+1}\cdot n^{\frac{3}{2}}}{(n+1)^{\frac{3}{2}}\cdot|-2|^n}} = \frac{1}{\lim\limits_{n\to\infty}\frac{2}{\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}} = \frac{1}{2}.$$

Consider the ending points $x = x_0 \pm R = \pm\frac{1}{2}$. The series $\sum_{n=1}^{\infty}\frac{[-2\cdot(-\frac{1}{2})]^n}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ is convergent by the p-series and the series $\sum_{n=1}^{\infty}\frac{(-2\cdot\frac{1}{2})^n}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\frac{3}{2}}}$ is convergent by the alternating series test. Thus the interval of convergence is $[-\frac{1}{2},\frac{1}{2}]$.

2 ii).

$$R = \frac{1}{\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \dfrac{3^{n+1}\cdot(n+1)}{(n+2)\cdot 3^n}} = \frac{1}{\lim\limits_{n\to\infty} \dfrac{3\cdot\left(1+\frac{1}{n}\right)}{1+\frac{2}{n}}} = \frac{1}{3}.$$

Consider the ending points $x = x_0 \pm R = 2 \pm \dfrac{1}{3}$. When $x = 2 - \dfrac{1}{3}$,

$$\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n+1} = \sum_{n=1}^{\infty} \frac{3^n\left(-\frac{1}{3}\right)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$

is convergent by the alternating series test. When $x = 2 + \dfrac{1}{3}$,

$$\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n+1} = \sum_{n=1}^{\infty} \frac{3^n\left(\frac{1}{3}\right)^n}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n+1}$$

is divergent by the p-series. Thus the interval of convergence is $\left[2-\dfrac{1}{3}, 2+\dfrac{1}{3}\right)$.

2 iii). Observe that

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n} \cdot \left(x - \frac{1}{3}\right)^n.$$

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{|-3|^n}{n}}} = \frac{1}{\lim\limits_{n\to\infty} \dfrac{3}{\sqrt[n]{n}}} = \frac{1}{3}.$$

Consider the ending points $x = x_0 \pm R = \dfrac{1}{3} \pm \dfrac{1}{3}$. When $x = 0$,

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent by the p-series. When $x = \frac{2}{3}$,

$$\sum_{n=1}^{\infty} \frac{(1-3x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

is convergent by the alternating series test. Thus the interval of convergence is $\left(0, \dfrac{2}{3}\right]$.

3 i). $f'(x) = 2e^{2x}$, $f''(x) = 2^2 e^{2x}$ and in general $f^{(k)}(x) = 2^k e^{2x}$. Thus

$$e^{2x} \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(3)}{k!}(x-3)^k = \sum_{k=0}^{\infty} \frac{2^k e^6}{k!}(x-3)^k = e^6 \sum_{k=0}^{\infty} \frac{2^k}{k!}(x-3)^k.$$

3 ii). Since $f(x) = \cos x$, we have

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = f(x) = \cos x.$$

In general,

$$f^{(k)}(x) = \begin{cases} \cos x & \text{if} \quad k = 4l \\ -\sin x & \text{if} \quad k = 4l+1 \\ -\cos x & \text{if} \quad k = 4l+2 \\ \sin x & \text{if} \quad k = 4l+3 \end{cases}$$

and so

$$f^{(k)}\left(\frac{\pi}{3}\right) = \begin{cases} \frac{1}{2} & \text{if} \quad k = 4l \\ -\frac{\sqrt{3}}{2} & \text{if} \quad k = 4l+1 \\ -\frac{1}{2} & \text{if} \quad k = 4l+2 \\ \frac{\sqrt{3}}{2} & \text{if} \quad k = 4l+3 \end{cases}$$

Thus

$$\cos x \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(\frac{\pi}{3})}{k!}\left(x - \frac{\pi}{3}\right)^k =$$

$$\sum_{l=0}^{\infty} \frac{\frac{1}{2}}{(4l)!}\left(x - \frac{\pi}{3}\right)^{4l} - \frac{\frac{\sqrt{3}}{2}}{(4l+1)!}\left(x - \frac{\pi}{3}\right)^{4l+1} - \frac{\frac{1}{2}}{(4l+2)!}\left(x - \frac{\pi}{3}\right)^{4l+2} + \frac{\frac{\sqrt{3}}{2}}{(4l+3)!}\left(x - \frac{\pi}{3}\right)^{4l+3}.$$