

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SPECIAL SEMESTER EXAMINATION 1999-2000

**MA2108    Advanced Calculus II**

June 2000 — Time allowed : 2 hours

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**INSTRUCTIONS TO CANDIDATES**

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions in **Section A**.
3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

**SECTION A**

Answer all the questions in this section. Section A carries a total of 60 marks.

**Question 1** [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

$$(i) \left\{ \frac{1}{\sqrt{n+1}(\sqrt{n+8}-\sqrt{n})} \right\}.$$

$$(ii) \left\{ \left( \frac{n}{n+3} \right)^{2n} \right\}.$$

$$(iii) \left\{ (n^2+1) \left( 1 - \cos \frac{1}{n} \right) \right\}.$$

$$(iv) \left\{ (4^n + 3^n)^{\frac{1}{2n}} \right\}.$$

**Question 2** [16 marks]

Determine the absolute convergence, conditional convergence or divergence of each of the following series. Justify your answers.

$$(i) \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}}.$$

$$(ii) \sum_{n=2}^{\infty} (-1)^n \frac{3 + \cos n}{n(\ln n)^{\frac{3}{2}}}.$$

$$(iii) \sum_{n=1}^{\infty} \frac{2^n (n!)^2}{(2n)!}.$$

$$(iv) \sum_{n=1}^{\infty} (-2)^n \left( 1 - \frac{1}{n} \right)^{n^2}.$$

**Question 3** [12 marks]

- (a) Find the radius of convergence of the power series  $\sum_{n=1}^{\infty} \frac{x^n}{n^2 + n}$ .
- (b) Find the interval of convergence of the power series  $\sum_{n=1}^{\infty} \frac{2^n (x - 1)^n}{n + 2}$ .

**Question 4** [16 marks]

- (a) Find the general solutions of the following differential equations:
- (i)  $y' - xe^y = 2e^y$ .
- (ii)  $y'' + 2y' + 3y = 0$ .
- (b) Evaluate:  $\lim_{y \rightarrow 0} \frac{\arctan y - \sin y}{y^3 \cos(y^2)}$ .

**SECTION B**

Answer not more than **TWO (2)** questions from this section. Each question in this section carries 20 marks.

**Question 5** [20 marks]

- (a) Evaluate  $\lim_{n \rightarrow \infty} \int_0^1 \left( \frac{x^3 + 1}{3} \right)^n \cos nx dx$ . Justify your answer.
- (b) Find the first three terms (up to and including the quadratic term) of the Taylor series for the function  $f(x) = \sqrt{x}$  at  $x = 100$ .
- (c) Show that the function

$$f(x) = \sum_{k=1}^{\infty} k^2 \sin^k x$$

is differentiable on  $(0, \frac{\pi}{2})$ .

**Question 6** [20 marks]

- (a) Let  $\sum_{k=1}^{\infty} a_k$  be an absolutely convergent series. Evaluate  $\int_0^{2\pi} \sum_{k=1}^{\infty} a_k \cos kx dx$ . Justify your answer.
- (b) Solve the following initial value problem:

$$y'' + 3y' + 2y = e^{-t}; \quad y(0) = 0, \quad y'(0) = 2.$$

- (c) A tank contains 100 gallons of salt water in which 40 pounds of salt is dissolved (so the concentration of salt is initially 0.4 pounds per gallon). It is desired to reduce the concentration to 0.1 pounds per gallon by pouring in 10 gallons of pure water per minute and allowing the (well-stirred, not shaken) mixture to flow out at the same rate. How long will this take?

**Question 7** [20 marks]

- (a) Find the Taylor series at  $x = 0$  for the function  $f(x) = \cos x^2$ .
- (b) Use an appropriate series to find a decimal approximation to  $\int_0^{0.1} \cos x^2 dx$  that is accurate to within  $10^{-6}$ . Be sure to explain why you know that your approximation is this accurate.
- (c) Prove that  $\sum_{k=0}^{\infty} \frac{x^k}{k!}$  does NOT converge uniformly on  $(-\infty, +\infty)$ .

**END OF PAPER**