

Optional Practice Questions for Chapter three

1. Find the pointwise limit $S(x)$ if it exists; then determine whether $S_n(x)$ converges uniformly to $S(x)$ on I .

- a) $S_n(x) = \frac{1}{nx+1}$ $I = [0, 1]$.
 b) $S_n(x) = x^n$ $I = (0, 1)$.
 c) $S_n(x) = n^2x(1-x^n)$ $I = [0, 1]$.
 d) $S_n(x) = \frac{1-x^n}{1+x^n}$ on $(0, 2)$.
 e) $S_n(x) = \frac{x^{2n}}{1+x^{2n}}$ on $[-2, 2]$.

2. Find the domain of the series of functions; then determine whether it converges uniformly on its domain.

- a) $\sum_{k=1}^{\infty} \left(\frac{1-x}{1+x} \right)^{k-1}$.
 b) $\sum_{k=1}^{\infty} \frac{1}{k} x^k$.
 c) $\sum_{k=1}^{\infty} \frac{1}{k^2} x^k$.
 d) $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$.
 e) $\sum_{k=1}^{\infty} (\cos k\pi) x^{2k}$.

3. Suppose that the series $\sum_{k=0}^{\infty} a_k$ converges. Show that the power series $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[0, 1]$.

Note. This is a special case of the Abel theorem. The Abel theorem is as follows.

If $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ has a positive radius of convergence R and if $\sum_{k=0}^{\infty} a_k R^k$ converges then $\sum_{k=0}^{\infty} a_k (x-x_0)^k$ converges uniformly on $[x_0, x_0+R]$. Similarly,

if $\sum_{k=0}^{\infty} a_k(-R)^k$ converges then $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ converges uniformly on $[x_0 - R, x_0]$.

4. By using Question 3 above show that

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{2k-1}.$$

5. Estimate the error E in writing

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + E$$

for x in the interval $[0, 0.5]$.

6. Show that

$$\sum_{k=1}^{\infty} k \cdot x^{2k-1} = \frac{x}{(1-x^2)^2}.$$

for $|x| < 1$. By using this, show that

$$\sum_{k=1}^{\infty} \frac{k}{2^{2k-1}} = \frac{1}{2} + \frac{2}{2^3} + \frac{3}{2^5} + \frac{4}{2^7} + \cdots = \frac{8}{9}.$$