Supplement to Chapter Two

Let $\sum_{n=1}^{\infty} a_n$ be a series of numbers. Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent. We are going to estimate the infinite sum $S = \sum_{n=1}^{\infty} a_n$.

Example. Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$. By taking the partial sum, we have

$$S \approx S_1 = 1,\ S \approx S_2 = 1+\frac{1}{2^2} = 1.25,\ S \approx S_3 = 1+\frac{1}{2^2}+\frac{1}{3^2} \approx 1.361, \ldots.$$

By using computer programs, we are able to compute much more, say $S_{1000000}$. A mathematical problem is then what is the 'error' for estimating S by using the partial sum S_n. In other words, how to estimate the remainder

$$R_n = |S - S_n| = |a_{n+1} + a_{n+2} + \cdots|.$$

Theorem [Integral Test Estimation]. *Let $f : [1,\infty) \to \mathbb{R}$ be a positive, continuous and monotone decreasing function. Suppose we have a series $\sum_{k=1}^{\infty} a_k$ such that $a_k = f(k)$, then the remainder*

$$R_n = |a_{n+1} + a_{n+2} + \cdots| \leq \int_n^{\infty} f(x)dx.$$

Proof. From the graph,

we see that

area of the rectangles $\leq$ area under $f(x)$ over $[n, \infty)$, that is,

$$R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)dx.$$

Example 1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1 + \frac{1}{2^2} + \ldots + \frac{1}{10^2}$ with the error

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2}dx = \frac{1}{10} = 0.1.$$

To get the estimation with error less than or equal to 0.01, we may need to add up the first 100 terms.

Example 2. *Estimate* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ *with error* ≤ 0.001.

From $\displaystyle\int_n^{\infty} \frac{1}{x^4}dx \leq 10^{-3}$, we have $\frac{1}{3}n^{-3} \leq 10^{-3}$ or $n^3 \geq \frac{10^3}{3}$ or $n \geq 7$. Thus

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \approx 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4}$$

with error ≤ 0.001.

Theorem [Alternating Series Estimation]. *Let* $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ *(or* $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$*) be an alternating series. Suppose that*

(i) $a_n > 0$ *for all* n,
(ii) a_n *is monotone decreasing* (*i.e.*, $a_n \geq a_{n+1}$ *for all* n), *and*
(iii) $\lim_{n \to 0} a_n = 0$.

Then the remainder

$$R_n = \left| \sum_{k=n+1}^{\infty} (-1)^{k+1} a_k \right| \le a_{n+1}$$

(resp. $R_n = \left| \sum_{k=n+1}^{\infty} (-1)^{k} a_k \right| \le a_{n+1}$*).*

Proof.

$$\begin{aligned} R_n &= \left|(-1)^{n+2} a_{n+1} + (-1)^{n+3} a_{n+2} + \cdots \right| \\ &= \left|a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots \right|. \end{aligned}$$

Since

$$a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots$$
$$= a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - (a_{n+6} - a_{n+7}) - \cdots \le a_{n+1}$$

and

$$a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + \cdots = (a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots \ge 0,$$

we have $R_n \le a_{n+1}$.

Example 3. *Estimate* $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$ *with error within* 0.001*.*

From $\frac{1}{(n+1)^4} \le 10^{-3}$, we have $n+1 \ge 6$ or $n \ge 5$. Thus

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4} \approx 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4}$$

with error within 0.001.