More Examples to Section 3.2

Let $\{F_n\}$ be a sequence of functions on an interval I. To see whether $\{F_n\}$ is uniformly convergent, we may do by the following steps.

(1). Determine the limiting function $F(x) = \lim\limits_{n\to\infty} F_n(x)$.

(2). Determine $T_n = \sup\limits_{x\in I} |F_n(x) - F(x)|$.

(3). Check whether $\lim\limits_{n\to\infty} T_n = 0$.

Note. The main point is to see whether $\lim\limits_{n\to\infty} T_n = 0$ or not.

Example 1. *Determine whether the following sequences of functions converge uniformly on the indicated interval.*

(a) $f_n(x) = \dfrac{n^2 \ln x}{x^n}$, $x \in [1, +\infty)$;

(b) $f_n(x) = \dfrac{n^2 \ln x}{x^n}$, $x \in [2, +\infty)$.

Solution. Let $f(x) = \lim\limits_{n\to\infty} f_n(x) = 0$ for $x \geq 1$.

(a). $T_n = \sup\limits_{x\geq 1} |f_n(x) - 0| = \sup\limits_{x\geq 1} \dfrac{n^2 \ln x}{x^n} = \sup\limits_{x\geq 1} f_n(x)$. From

$$f_n'(x) = n^2 \frac{1}{x} \cdot x^{-n} - n^3 \ln x \cdot x^{-n-1} = \frac{n^2 - n^3 \ln x}{x^{n+1}} = 0,$$

we have $n^2 - n^3 \ln x = 0$ or $x = e^{\frac{1}{n}}$. Observe that $f_n(x)$ is monotone increasing for $1 \leq x \leq e^{\frac{1}{n}}$ and monotone decreasing for $x \geq e^{\frac{1}{n}}$. Thus

$$T_n = \max_{x\geq 1} f_n(x) = f_n(e^{\frac{1}{n}}) = \frac{n^2 \cdot \frac{1}{n}}{\left(e^{\frac{1}{n}}\right)^n} = \frac{n}{e} \not\to 0$$

as $n \to \infty$ and so $\{f_n(x)\}$ does NOT converge uniformly.

(b). Since $e^{\frac{1}{n}} \leq 2$ for $n \geq 2$, the function $f_n(x)$ is monotone decreasing on $[2, +\infty)$ for $n \geq 2$ and so $T_n = \sup\limits_{x\geq 2} |f_n(x) - f(x)| = f_n(2) =$

$\dfrac{n^2 \ln 2}{2^n}$ for $n \geq 2$. Since $\lim_{n\to\infty} T_n = 0$, $\{f_n\}$ converges uniformly on $[2, +\infty)$. $\square$

Sometimes we do not need to figure out T_n EXPLICITLY.

Example 2. *Show that* $F_n(x) = \dfrac{n^2 \ln x \sin nx}{x^n}$ *converges uniformly on* $[2, +\infty)$*.*

Solution. $F(x) = \lim_{n\to\infty} F_n(x) = 0$ for $x \geq 2$. Observe

$$T_n = \sup_{x\geq 2} |F_n(x) - F(x)| = \sup_{x\geq 2} \frac{n^2 \ln x |\sin nx|}{x^n} \leq \frac{n^2 \ln 2}{2^n}$$

for $n \geq 2$. Since $\lim_{n\to\infty} T_n = 0$, $\{F_n\}$ converges uniformly. $\square$