Taylor Formula and its Applications

Theorem. *If f and its first $(n+1)$ derivatives f', f'', $\cdots$, $f^{(n+1)}$ are continuous on $[x_0, x]$ or $[x, x_0]$, then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x),$$

where $R_n(x) = \dfrac{1}{n!}\displaystyle\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt.$

Proof. We may assume that $x_0 < x$. Let

$$g(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

be a function on t for $x_0 \le t \le x$. Then

$$\begin{aligned} g'(t) &= -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!} k(x-t)^{k-1} \cdot (-1) \\ &\quad -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \\ &= -\left[\frac{f'(t)}{0!} + \frac{f''(t)}{1!}(x-t) + \cdots + \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] \\ &\quad + \left[\frac{f'(t)}{0!} + \frac{f''(t)}{1!}(x-t) + \cdots + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^(n-1)\right] \\ &= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n. \end{aligned}$$

Thus $g(x) - g(x_0) = \displaystyle\int_{x_0}^{x} g'(t)dt = -\int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt = -R_n(x)$ and so

$$f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-x_0)^k = g(x_0) = R_n(x)$$

because $g(x) = 0$. □

Lemma. *Let f and g be continuous on $[a,b]$. Suppose that g does not change sign in $[a,b]$. Then there is a point $\xi \in (a,b)$ such that*

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx.$$

Proof. We may assume that $g(x) \geq 0$ for $x \in [a,b]$. Since f is continuous on $[a,b]$, f has maximum and minimum on $[a,b]$. Let $M = \max\{f(x)|a \leq x \leq b\}$ and $m = \min\{f(x)|a \leq x \leq b\}$. Then $m \leq f(x) \leq M$ for $a \leq x \leq b$ and so

$$m\int_a^b g(x)dx = \int_a^b mg(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx.$$

If $\displaystyle\int_a^b g(x)dx = 0$, then $\int_a^n f(x)g(x)dx = 0$ and the assertion holds in this case. Otherwise $\displaystyle\int_a^b g(x)dx > 0$ and so

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M.$$

By the intermediate-value theorem, there is a point ξ between a and b such that

$$f(\xi) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}.$$

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Taylor Formula. *If f has derivatives of all orders in an open interval I containing x_0, then for each positive integer n and for each x in I,*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x),$$

where the remainder $R_n(x) = \frac{f^{(n+1}(\xi))}{(n+1)!}(x-x_0)^{n+1}$.

Proof. Since $g(t) = (x-t)^n$ does not change the sign on $[x_0, x]$, by the above lemma, we have

$$\begin{aligned} R_n(x) &= \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^n dt = \frac{f^{(n+1)}(\xi)}{n!}\int_{x_0}^x (x-t)^n dt \\ &= \frac{f^{(n+1)}(\xi)}{n!}\frac{(x-x_0)^{n+1}}{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

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Note. The Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ converges to $f(x)$, that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k,$$

if and only if the Taylor remainder $R_n(x) \to 0$ as $n \to \infty$.

Taylor Estimation. If f has derivatives of all orders in an open interval I containing x_0, then for each positive integer n and for each x in I,

$$f(x) \approx T_n(f) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

with error

$$|R_n(x)| \leq \frac{\max_{t\in[x_0,x]}\{f^{(n+1}(t))\}}{(n+1)!}|x-x_0|^{n+1}$$

Example 1. *Show that* $e^x = \sum_{k=0}^{\infty} \frac{1}{n!}x^k$ *for* $x \in (-\infty, +\infty)$.

Solution. Let $f(x) = e^x$ and $x_0 = 0$. Then $f^{(k)}(x) = e^x$ and so $f^{(k)}(0) = 1$ for all $k \geq 0$. Since the Taylor remainder

$$|R_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}\right| \leq \left|\frac{\max\{e^x, 1\} \cdot x^{n+1}}{(n+1)!}\right| \to 0$$

as $n \to \infty$ for any given x. Thus e^x equals to its Taylor series for any given x, that is,

$$e^x = \sum_{k=0}^{\infty} \frac{1}{n!}x^k = 1 + x + \frac{x^2}{2!} + \cdots$$

□

Applications

Binomial Series. Let m be any constant. Then

$$\begin{aligned}(1+x)^m &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots \\ &= 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k = \sum_{k=0}^{\infty} \binom{m}{k} x^k\end{aligned}$$

for $|x| < 1$, where $\binom{m}{0} = 1$, $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2!}$ and

$$\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

for $k \geq 3$.

Example 2.

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \cdots$$

$$\sqrt{1-x^3} = (1-x^3)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(-x^3\right)^k = 1 - \frac{1}{2}x^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^6 + \cdots$$

$$\sqrt{4.1} = \left(4 + \frac{1}{10}\right)^{\frac{1}{2}} = 2\left(1 + \frac{1}{40}\right)^{\frac{1}{2}} = 2\sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{1}{40}\right)^k.$$

Evaluating Integrals.

Example 3. *Evaluate the integral* $\int_0^1 \sin x^2 dx$ *with an error of less than to* 0.001.

Solution. From $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, we have

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{4k-2}}{(2k-1)!}$$

and so

$$\int_0^1 \sin x^2 dx = \int_0^1 x^2 dx - \int_0^1 \frac{x^6}{3!} dx + \int_0^1 \frac{x^{10}}{5!} dx + \cdots$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \int_0^1 \frac{x^{4k-2}}{(2k-1)!} dx = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(2k-1)!(4k-1)}$$

is an alternating series. Let $a_n = \frac{1}{(2n-1)!(4n-1)}$. From

$$a_{n+1} = \frac{1}{(2n+1)!(4n+3)} < \frac{1}{1000},$$

we have $(2n+1)!(4n+3) > 1000$ or $n \geq 3$. Thus

$$\int_0^1 \sin x^2 dx \approx \frac{1}{1! \cdot 3} - \frac{1}{3! \cdot 7} + \frac{1}{5! \cdot 11}$$

with error less that 0.001. $\square$

Example 4. For $|x| < 1$, we have the Taylor series

$$\arctan x = \int_0^x \frac{1}{1+t^2}dt = \int_0^x \sum_{k=0}^{\infty}(-1)^k t^{2k}dt$$
$$= \sum_{k=0}^{\infty}(-1)^k \int_0^x t^{2k}dt = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$$

for $|x| < 1$. This formula also holds for the ending points $x = \pm 1$ (We omit the proof of this!), that is,

$$\arctan x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$$

for $|x| \leq 1$ and so

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots .$$

Evaluating Limits.
Example 5.

$$\lim_{x\to 0} \frac{\sin(x^2) - \arctan(x^2)}{x^6}$$
$$= \lim_{x\to 0} \frac{(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots) - (x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \cdots)}{x^6}$$
$$= \lim_{x\to 0} \left(-\frac{1}{3!} + \frac{1}{3}\right) + \left(\frac{1}{5!} - \frac{1}{5}\right)x^4 + \cdots = -\frac{1}{3!} + \frac{1}{3} = \frac{1}{6}.$$

Example 6. Let $f(x) = \sqrt[3]{1+x^2}$. Find $f^{(10)}(0)$.

Solution. From

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(x) = (1+x^2)^{\frac{1}{3}} = \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} \left(x^2\right)^k$$
$$= \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} x^{2k}$$

for $|x| < 1$, we have $\frac{f^{(10)}(0)}{10!} = \binom{\frac{1}{3}}{5}$ by comparing the coefficients of x^{10} and so

$$f^{(10)}(0) = 10! \cdot \binom{\frac{1}{3}}{5} =$$

□