## Supplement to Section 4.6

We are going to find a formula for determining a particular solutions $y_p(t)$ of the differential equation

$$ay'' + by' + cy = g(t) \tag{1}$$

with $a \neq 0$. Recall that the associated homogeneous equation is

$$ay'' + by' + cy = 0. \tag{2}$$

Let $y_h(t) = Ay_1(t) + By_2(t)$ be the general solution of the equation (2). We try

$$y_p(t) = A(t)y_1(t) + B(t)y_2(t) \tag{3}$$

for *undetermined* functions $A(t)$ and $B(t)$ satisfying

$$A'(t)y_1(t) + B'(t)y_2(t) = 0 \qquad (4).$$

**Note.** When we plug $y_p$ into equation (1), we obtain an equation for the functions $A(t)$ and $B(t)$. Technically we make equation (4) as an additional equation such that we will be able to find $A(t)$ and $B(t)$.

Now we compute $y_p'$ and $y_p''$.

$$y_p' = (A'y_1 + B'y_2) + (Ay_1' + By_2') = Ay_1' + By_2' \tag{5}$$

$$y_p'' = A'y_1' + B'y_2' + Ay_1'' + By_2'' \tag{6}$$

From equation (1), we have

$$g(t) = a(A'y_1' + B'y_2' + Ay_1'' + By_2'') + b(Ay_1' + By_2') + c(Ay_1 + By_2)$$

$$= aA'y_1' + aB'y_2' + A(ay_1'' + by_1' + cy_1') + B(ay_2'' + by_2' + cy_2) = aA'y_1' + aB'y_2'.$$

Thus we have the equation

$$\begin{cases} aA'(t)y_1'(t) + aB'(t)y_2'(t) = g(t) \\ A'(t)y_1(t) + B'(t)y_2(t) = 0 \end{cases}$$

It follows that

$$\begin{cases} A'(t) = \dfrac{\begin{vmatrix} g(t) & ay_2'(t) \\ 0 & y_2(t) \end{vmatrix}}{\begin{vmatrix} ay_1'(t) & ay_2'(t) \\ y_1(t) & y_2(t) \end{vmatrix}} = \dfrac{y_2(t)g(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)} \\ B'(t) = \dfrac{\begin{vmatrix} ay_1'(t) & g(t) \\ y_1(t) & 0 \end{vmatrix}}{\begin{vmatrix} ay_1'(t) & ay_2'(t) \\ y_1(t) & y_2(t) \end{vmatrix}} = \dfrac{-y_1(t)g(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)} \end{cases}$$

and so we have the formula

$$y_p(t) = \left(\int \frac{g(t)y_2(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right) \cdot y_1(t) + \left(\int \frac{-y_1(t)g(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right) \cdot y_2(t)$$

**Example 1.** Solve $y'' + y = t^2$.

*Solution.* From $y'' + y = 0$, we have $r^2 + 1 = 0$ or $r = \pm i$.

$$y_h = A\cos t + B\sin t,$$

where $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

$$\Delta = \begin{vmatrix} ay_1'(t) & ay_2'(t) \\ y_1(t) & y_2(t) \end{vmatrix} = \begin{vmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1.$$

$$\begin{aligned} y_p(t) &= \left(\int \frac{g(t)y_2(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right) \cdot y_1(t) + \left(\int \frac{-y_1(t)g(t)}{ay_1'(t)y_2(t) - ay_1(t)y_2'(t)}dt\right) \cdot y_2(t) \\ &= \left(\int \frac{t^2\sin t}{-1}dt\right)\cos t + \left(\int \frac{-t^2\cos t}{-1}dt\right)\sin t \end{aligned}$$

Observe that

$$\int -t^2 \sin t dt = \int t^2 d\cos t = t^2\cos t - \int 2t\cos t dt = t^2\cos t - 2\int t d\sin t$$

$$= t^2 \cos t - 2t \sin t + 2 \int \sin t dt = t^2 \cos t - 2t \sin t - 2 \cos t$$

$$\int t^2 \cos t dt = \int t^2 d \sin t = t^2 \sin t - \int 2t \sin t dt = t^2 \sin t + 2 \int t d \cos t$$

$$= t^2 \sin t + 2t \cos t - 2 \int \cos t dt = t^2 \sin t + 2t \cos t - 2 \sin t$$

Thus

$$y_p(t) = (t^2 \cos t - 2t \sin t - 2 \cos t) \cos t + (t^2 \sin t + 2t \cos t - 2 \sin t) \sin t = t^2 - 2$$

and so

$$y(t) = A \cos t + B \sin t + t^2 - 2.$$

□

**Example 2.** Solve $y'' + y = \dfrac{1}{\cos t}$.

*Solution.* From $y'' + y = 0$, we have $r^2 + 1 = 0$, $r = \pm i$ and so $y_h = A \cos t + B \sin t$, where $y_1(t) = \cos t$ and $y_2(t) = \sin t$. As above,

$$\Delta = \begin{vmatrix} ay_1'(t) & ay_2'(t) \\ y_1(t) & y_2(t) \end{vmatrix} = \begin{vmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1.$$

and so

$$y_p(t) = \left( \int \frac{g(t) y_2(t)}{ay_1'(t) y_2(t) - ay_1(t) y_2'(t)} dt \right) \cdot y_1(t) + \left( \int \frac{-y_1(t) g(t)}{ay_1'(t) y_2(t) - ay_1(t) y_2'(t)} dt \right) \cdot y_2(t)$$

$$= \left( \int -\frac{\sin t}{\cos t} dt \right) \cos t + \left( \int \frac{\cos t}{\cos t} dt \right) \sin t = (\ln |\cos t|) \cos t + t \sin t$$

and so

$$y(t) = A \cos t + B \sin t + (\ln |\cos t|) \cos t + t \sin t$$

□