1. Denote the set of rational numbers by $\mathbb{Q}$. Consider the set
$$S = \{x \in \mathbb{Q} | 0 \leq x < 1\}.$$
Find $\sup S$ and $\inf S$. Justify your answers.
2. Let A and B be two non-empty bounded set of real numbers such that $A \subseteq B$. Show that $\inf A \geq \inf B$.
3. Let A and B be two non-empty bounded set of real numbers
   i) Show that $\sup A \cup B = \max\{\sup A, \sup B\}$.
   ii) Is it true that $\sup A \cap B = \min\{\sup A, \sup B\}$? Justify your answer.
4. Consider the sequence $\{a_n\}$ defined recursively by
$$a_1 = 2, \quad a_n = \sqrt{6 + a_{n-1}}, \quad n = 2, 3, 4, \cdots .$$
   i) Show that $2 \leq a_n \leq 3$ for all n.
   ii) Show that $\{a_n\}$ is monotone increasing.
   iii) Using parts i) and ii), show that $\{a_n\}$ converges, and find its limit.
5. Consider the sequence $\{x_n\}$ defined recursively by
$$x_1 = \tfrac{3}{4}, \quad x_{n+1} = 2x_n - x_n^2, \quad n = 1, 2, 3, \cdots .$$
Show that $\{x_n\}$ converges, and find its limit. (Hint: Show that $x_n \leq 1$ for all n and $\{x_n\}$ is monotone increasing.)
6. Let $\{a_n\}$ be a convergent sequence. Show that $\overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} a_n$
7. Find the lim sup and lim inf of the sequences:
   (a). $\{4 + \cos \frac{n\pi}{2}\}$.
   (b). $\{\frac{1+(-1)^n}{n}\}$.
8. Let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequence of real numbers. Prove that $\{a_n b_n\}$ is also a Cauchy sequence.

Some suggested answers:

1. $\sup S = 1$ and $\inf S = 0$.
4. $\lim_{n\to\infty} a_n = 3$.
5. $\lim_{n\to\infty} x_n = 1$.
7. a) $\limsup = 5$ and $\liminf = 3$.
7. b) $\limsup = \liminf = \lim = 0$.

You should try to prove Questions 2 and 3 by yourself first. Below I give a solution of Question 4 and then you should try to solve Question 5. In Question 6, I show that $\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} a_n$ and then you try to prove that $\underline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} a_n$. You should try to solve Question 8 by yourself first and then compare your solution with the answer given below.

Solution to Question 4: (i) We prove that $2 \leq a_n \leq 3$ by induction on n. Since $a_1 = 2$, we have $2 \leq a_1 \leq 3$. Suppose that $a_{n-1} \leq 3$ with $n \geq 2$. Then
$$2 \leq \sqrt{6+2} \leq a_n = \sqrt{6 + a_{n-1}} \leq \sqrt{6+3} = 3.$$

The induction is finished and hence the statement.
(ii) Let $n \geq 2$. Then

$$a_n - a_{n-1} = \sqrt{6 + a_{n-1}} - a_{n-1} = \frac{(\sqrt{6 + a_{n-1}} - a_{n-1})(\sqrt{6 + a_{n-1}} + a_{n-1})}{\sqrt{6 + a_{n-1}} + a_{n-1}}$$

$$= \frac{6 + a_{n-1} - a_{n-1}^2}{\sqrt{6 + a_{n-1}} + a_{n-1}} \geq 0$$

because $\sqrt{6 + a_{n-1}} + a_{n-1} > 0$ and $6 + x - x^2 = -(x-3)(x+2) \geq 0$ for $-2 \leq x \leq 3$. Thus $\{a_n\}$ is monotone increasing.
(iii) By (i) and (ii), $\{a_n\}$ is bounded above and monotone increasing. Thus $\{a_n\}$ is convergent. Let $A = \lim_{n\to\infty} a_n$. Then we have the equation

$$A = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{6 + a_{n-1}} = \sqrt{6 + \lim_{n\to\infty} a_{n-1}} = \sqrt{6 + A}$$

and so $A^2 = 6 + A$. It follows that $A = -2$ or 3. Since $a_n \geq 2$ for each n, $A = \lim_{n\to\infty} a_n \geq 2$ and so $A = 3$.

Partial Solution of Question 6: We prove that $\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} a_n$. Let $A = \lim_{n\to\infty} a_n$ and let $b_n = \sup\{a_n, a_{n+1}, \cdots\}$. Given any $\epsilon > 0$, there is an integer N such that $|a_n - A| < \epsilon/2$ for $n > N$. Thus

$$A - \epsilon/2 < a_n, a_{n+1}, a_{n+2}, \ldots < A + \epsilon/2$$

for $n > N$ and so $A + \epsilon$ is an upper bound of the set $\{a_n, a_{n+1}, \ldots\}$ for $n > N$. Since b_n is the least upper bound of $\{a_n, a_{n+1}, \ldots\}$, we have

$$A - \epsilon/2 < b_n = \sup\{a_n, a_{n+1}, \ldots\} \leq A + \epsilon/2$$

or $|b_n - A| \leq \epsilon/2 < \epsilon$ for $n > N$. Thus

$$\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} b_n = A = \lim_{n\to\infty} a_n.$$

Solution of Question 8. Since $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences, both $\{a_n\}$ and $\{b_n\}$ are bounded. Thus there exist positive numbers $M_1 > 0$ and $M_2 > 0$ such that $|a_n| \leq M_1$ and $|b_n| \leq M_2$ for all n. Given $\epsilon > 0$, since $\{a_n\}$ is a Cauchy sequence, there is a positive integer N_1 such that $|a_n - a_m| < \epsilon/M_2$ for all $n, m > N_1$ and, since $\{b_n\}$ is a Cauchy sequence, there is a positive integer N_2 such that $|b_n - b_m| < \epsilon/M_1$ for all $n, m > N_2$. Let $N = \max\{N_1, N_2\}$. When $n, m > N$, we have

$$|a_n b_n - a_m b_m| = |(a_n b_n - a_n b_m) + (a_n b_m - a_m b_m)| \leq |a_n b_n - a_n b_m| + |a_n b_m - a_m b_m|$$

$$= |a_n||b_n - b_m| + |a_n - a_m||b_m| \leq M_1 \cdot |b_n - b_m| + M_2 \cdot |a_n - a_m| < M_1 \cdot \frac{\epsilon}{M_1} + M_2 \cdot \frac{\epsilon}{M_2} = \epsilon$$

and so $\{a_n b_n\}$ is a Cauchy sequence.