

1. (a). Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with each $a_n \geq 0$. Show that the series $\sum_{n=1}^{\infty} a_n^2$ is also convergent.
 [Hint: First show that $|a_n| < 1$ for n sufficiently large, and then use the comparison test.]
- (b). Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ such that $\sum_{n=1}^{\infty} \sqrt{a_n}$ diverges.
2. Use the integral test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)}$.
3. Find the domain of the functions
- (a). $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^x}$.
- (b). $g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^x + 1}$.
4. Use the ratio test to determine the convergence or divergence of each of the following series.
- (a). $\sum_{n=1}^{\infty} \frac{(3n)!}{6^n n! (2n)!}$
- (b). $\sum_{n=1}^{\infty} a_n$, where $a_1 = 1$, $a_n = 2 \left(1 - \frac{1}{n}\right)^n a_{n-1}$, $n = 2, 3, \dots$.
5. Use the (simplified) root test to determine the convergence or divergence of each of the following series.
- (a). $\sum_{n=1}^{\infty} \frac{5n^2 \cdot 3^n}{4^{n+4}}$.
- (b). $\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n} \left(1 - \frac{1}{2n}\right)^{n^2}$.
- (c). $\frac{1}{4} + \frac{1}{5^2} + \frac{1}{4^3} + \frac{1}{5^4} + \frac{1}{4^5} + \frac{1}{5^6} + \frac{1}{4^7} + \frac{1}{5^8} + \dots$.
6. Determine the convergence or divergence of each of the following series. Justify your answers.
- (a). $\sum_{n=1}^{\infty} (\sqrt{2n+2} - \sqrt{n})$.

$$(b). \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \cdot \frac{2^n}{5^n}.$$

$$(c). \sum_{n=1}^{\infty} \frac{\ln n}{n^{1.2}}.$$

$$(d). \sum_{n=1}^{\infty} \left(\frac{n}{n+2} \right)^{n^2}.$$

$$(e). \sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}.$$

$$(f). \sum_{n=1}^{\infty} \left(\frac{4}{9} + \frac{n^3}{3^n} \right)^{\frac{n}{2}}.$$

Some suggested answers:

2. divergent.

3(a). $(1, \infty)$.

3(b). $(0, \infty)$.

4(a). divergent.

4(b). convergent.

5(a). convergent.

5(b). divergent.

5(c). convergent.

6(a). divergent.

6(b). convergent.

6(c). convergent.

6(d). convergent.

6(e). divergent.

6(f). convergent.

Solution to Question 1(a). Since $\sum_{n=1}^{\infty} a_n$ is convergent, we have $\lim_{n \rightarrow \infty} a_n = 0$ and so there exists a positive integer N such that $a_n = |a_n| = |a_n - 0| < 1$ for $n > N$. It follows

$$a_n^2 = a_n \cdot a_n \leq 1 \cdot a_n = a_n$$

for $n > N$. By the comparison test, the positive series $\sum_{n=1}^{\infty} a_n^2$ is convergent.

Solution to Question 3(a). Consider the function

$$f(y) = \frac{1}{y(1 + \ln y)^x} = y^{-1}(1 + \ln y)^{-x}$$

on y for $1 \leq y < \infty$. Then $f(y)$ is a positive function. Since

$$f'(y) = -y^{-2}(1 + \ln y)^{-x} + y^{-1}(-x)(1 + \ln y)^{-x-1}y^{-1} = -y^{-2}(1 + \ln y)^{-x}\left(1 + \frac{x}{1 + \ln y}\right) \leq 0$$

for $y \geq \max\{1, e^{-x-1}\}$, the function $f(y)$ is eventually monotone decreasing. Observe that the integral

$$\begin{aligned} \int f(y)dy &= \int \frac{1}{y(1 + \ln y)^x} dy \\ &= \int z^{-x} dz = \begin{cases} = \frac{1}{-x+1} z^{-x+1} = \frac{1}{-x+1} (1 + \ln y)^{-x+1} & \text{if } x \neq 1 \\ = \ln z = \ln(1 + \ln y) & \text{if } x = 1, \end{cases} \end{aligned}$$

where $z = 1 + \ln y$.

Case I. $x < 1$ or $-x + 1 > 0$. Since

$$\int_1^n f(y)dy = \frac{1}{-x+1} (1 + \ln y)^{-x+1} \Big|_1^n = \frac{1}{-x+1} (1 + \ln n)^{-x+1} - \frac{1}{-x+1},$$

the integral $\int_1^{\infty} f(y)dy$ is divergent and so the series $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^x}$ is divergent by

the integral test. In other words, the function $\zeta(x)$ is not defined for $x < 1$.

Case II. $x = 1$. Since

$$\int_1^n f(y)dy = \ln(1 + \ln n),$$

the integral $\int_1^{\infty} f(y)dy$ is divergent and so the series $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)}$ is divergent by

the integral test. In other words, the function $\zeta(x)$ is not defined at $x = 1$.

Case III. $x > 1$ or $-x + 1 < 0$. Since

$$\int_1^n f(y)dy = \frac{1}{-x+1} (1 + \ln y)^{-x+1} \Big|_1^n = \frac{1}{-x+1} (1 + \ln n)^{-x+1} - \frac{1}{-x+1},$$

the integral $\int_1^{\infty} f(y)dy$ is convergent to $0 - \frac{1}{-x+1} = \frac{1}{x-1}$ and so the series

$\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^x}$ is convergent by the integral test. Thus the function $\zeta(x)$ is well-defined for $x > 1$.

In conclusion, the domain of the function $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^x}$ is the infinite open interval $(1, +\infty)$.