MA2108 **Midterm** **1 October 2002**
Professor J. Wu

Time allowed: 1.5 hours

Tutorial Group:(circle one)

Tuesday 4-6 Wednesday 9-10 Wednesday 10-11 Wednesday 11-12

Wednesday 12-1 Friday 12-1 Friday 1-2

Matric number:____

Name:____

Signature:____

MA2108 Midterm **1 October** **Name:______________**

Problem #	Your Grades
1 (10 points)	
2 (10 points)	
3 (10 points)	
4 (10 points)	
5 (10 points)	
6 (10 points)	
7 (10 points)	
8 (10 points)	
9 (10 points)	
10 (10 points)	
total (100 points)	

Determine the limit of the following sequences:

1. $\left\{\cos\left(\dfrac{n+n^2\pi+2}{6+n+3n^2}\right)+\ln\left(\dfrac{3n+n^2+1}{4+n^2+2n}\right)\right\}.$

Solution.

$$\begin{aligned}
&\lim_{n\to\infty}\cos\left(\frac{n+n^2\pi+2}{6+n+3n^2}\right)+\ln\left(\frac{3n+n^2+1}{4+n^2+2n}\right)\\
&=\lim_{n\to\infty}\cos\left(\frac{1/n+\pi+2/n^2}{6/n^2+1/n+3}\right)+\ln\left(\frac{3/n+1+1/n^2}{4/n^2+1+2/n}\right)\\
&=\cos\frac{\pi}{3}+\ln 1=\frac{1}{2}.
\end{aligned}$$

$\square$

2. $\left\{\dfrac{\ln n+n^{60}60^n+n!}{3^n+n!}\right\}.$

Solution.

$$\lim_{n\to\infty}\frac{\ln n+n^{60}60^n+n!}{3^n+n!}=\lim_{n\to\infty}\frac{\dfrac{\ln n}{n!}+\dfrac{n^{60}}{2^n}\dfrac{120^n}{n!}+1}{\dfrac{3^n}{n!}+1}=\frac{0+0+1}{0+1}=1.$$

$\square$

3. $\left\{\sqrt{n}(\sqrt{n+2}-\sqrt{n})\right\}$.

Solution.

$$\lim_{n\to\infty}\sqrt{n}(\sqrt{n+2}-\sqrt{n})=\lim_{n\to\infty}\frac{\sqrt{n}(n+2-n)}{\sqrt{n+2}+\sqrt{n}}=\lim_{n\to\infty}\frac{2}{\sqrt{1+2/n}+1}=\frac{2}{1+1}=1.$$

$\square$

4. $\left\{\left(1-\dfrac{2}{n+3}\right)^{2n+\ln n}\right\}$.

Solution.

$$\lim_{n\to\infty}\left(1-\frac{2}{n+3}\right)^{2n+\ln n}=\lim_{n\to\infty}\left[\left(1-\frac{2}{n+3}\right)^{n+3}\right]^{\dfrac{2n+\ln n}{n+3}}$$

$$=\lim_{n\to\infty}\left[\left(1-\frac{2}{n+3}\right)^{n+3}\right]^{\dfrac{2+\ln n/n}{1+3/n}}=\left(e^{-2}\right)^{\dfrac{2+0}{1+0}}=e^{-4}.$$

$\square$

Determine convergence or divergence of the following series:

5. $\displaystyle\sum_{n=1}^{\infty}\frac{2\ln n+4n^2+5n-2}{2n^2+n^4+1}$.

Solution. Let $a_n=\dfrac{2\ln n+4n^2+5n-2}{2n^2+n^4+1}$ and $b_n=\dfrac{1}{n^2}$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{2\ln n+4n^2+5n-2}{2n^2+n^4+1}\cdot n^2=\lim_{n\to\infty}\frac{\dfrac{2\ln n}{n^2}+4+\dfrac{5}{n}-\dfrac{2}{n^2}}{\dfrac{2}{n^2}+1+\dfrac{1}{n^4}}=4.$$

Since the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-series, the series $\displaystyle\sum_{n=1}^{\infty}\frac{2\ln n+4n^2+5n-2}{2n^2+n^4+1}$ is convergent by the limit comparison test. $\square$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{3^n\cdot n!}$.

Solution. Let $a_n = \dfrac{n^n}{3^n \cdot n!}$. Since

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^{n+1}\cdot 3^n \cdot n!}{3^{n+1}\cdot (n+1)!\cdot n^n} \\ &= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n \cdot (n+1)}{3\cdot(n+1)} = \frac{e}{3} < 1,\end{aligned}$$

the series is convergent by the ratio test. ☐

Determine the absolute convergence, conditional convergence or divergence of the following series:

7. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\ln n - 2}{\sqrt{n}}$.

Solution. Let $a_n = \dfrac{\ln n - 2}{\sqrt{n}}$. Then $a_n > 0$ for $\ln n > 2$ or $n > e^2$. Let $f(x) = \dfrac{\ln x - 2}{\sqrt{x}}$. Then

$$f'(x) = \frac{\frac{1}{x}\sqrt{x} - \frac{\ln x - 2}{2\sqrt{x}}}{(\sqrt{x})^2} = \frac{3-\ln x}{2x\sqrt{x}} \leq 0$$

when $\ln x \geq 3$ or $x \geq e^3$. Thus $\{a_n\}$ is (eventually) monotone decreasing. Moreover

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} - \frac{2}{\sqrt{n}} = 0 - 0 = 0.$$

The series is convergent by the alternating series test. Now

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1}\frac{\ln n - 2}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{|\ln n - 2|}{\sqrt{n}}.$$

Since

$$\frac{|\ln n - 2|}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}$$

for $\ln n \geq 3$ or $n \geq e^3$ and the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by the p-series, the series $\displaystyle\sum_{n=1}^{\infty} \left| (-1)^{n+1}\frac{\ln n - 2}{\sqrt{n}} \right|$ is divergent by the comparison test and hence the original series is conditionally convergent. ☐

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin n}{n\left((\ln n)^2 + 1\right)}$.

Solution. Observe that $\left|\dfrac{(-1)^{n+1}\sin n}{n\left[(\ln n)^2+1\right]}\right| \leq \dfrac{1}{n\left[(\ln n)^2+1\right]}$. Let $f(x) = \dfrac{1}{x\left[(\ln x)^2+1\right]}$. Then $f(x)$ is positive and monotone decreasing. Since the integral

$$\int_1^{\infty} f(x)\ dx = \int_1^{\infty} \frac{1}{x\left[(\ln x)^2+1\right]}\ dx \xlongequal[dy=\frac{dx}{x}]{y=\ln x} \int_0^{\infty} \frac{1}{y^2+1}\ dy = \arctan y\Big|_0^{\infty} = \frac{\pi}{2}$$

converges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\left[(\ln n)^2+1\right]}$ is convergent. By the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}\sin n}{n\left((\ln n)^2+1\right)}\right|$ is convergent and hence the original series is absolutely convergent.

□

9. Find limit inferior and limit superior of the sequences:

$$\left\{\left(1+\frac{2}{n}\right)\left(1+\cos\frac{n\pi}{6}\right)^{\dfrac{1}{n}}\right\}$$

Solution. Let $n = 12k + r$, where k and r are integers with $r = 0, 1, \cdots, 11$ and $k \geq 0$. Then

$$1+\cos\frac{n\pi}{6} = 1+\cos(2k\pi+\frac{r\pi}{6}) = 1+\cos\frac{r\pi}{6}.$$

If $r \neq 6$, then $1+\cos\dfrac{r\pi}{6} \geq 1-\dfrac{\sqrt{3}}{2}$, and so, for $r \neq 6$,

$$\left(1+\frac{2}{12k+r}\right)\cdot\left(1-\frac{\sqrt{3}}{2}\right)^{\dfrac{1}{12k+r}}$$

$$\leq \left(1+\frac{2}{12k+r}\right)\left(1+\cos\frac{(12k+r)\pi}{6}\right)^{\dfrac{1}{12k+r}} \leq \left(1+\frac{2}{12k+r}\right)\cdot 2^{\frac{1}{12k+r}}.$$

Since

$$\lim_{k\to\infty}\left(1+\frac{2}{12k+r}\right)\cdot\left(1-\frac{\sqrt{3}}{2}\right)^{\frac{1}{12k+r}} = 1\cdot\left(1-\frac{\sqrt{3}}{2}\right)^{0} = 1$$

$$\lim_{k\to\infty}\left(1+\frac{2}{12k+r}\right)\cdot 2^{\frac{1}{12k+r}} = 1\cdot 2^0 = 1,$$

$$\lim_{k\to\infty}\left(1+\frac{2}{12k+r}\right)\left(1+\cos\frac{(12k+r)\pi}{6}\right)^{\dfrac{1}{12k+r}} = 1$$

for $r \neq 6$ by the Squeeze Theorem. When $r = 6$,

$$\lim_{k\to\infty} \left(1 + \frac{2}{16k + r}\right) \left(1 + \cos\frac{(12k + r)\pi}{6}\right)^{\dfrac{1}{12k + r}}$$

$$= \lim_{k\to\infty} \left(1 + \frac{2}{12k + r}\right) (1 - 1)^{\dfrac{1}{12k + r}} = \lim_{k\to\infty} 0 = 0.$$

Thus the set of subsequential limits is $\{0, 1\}$ and so

$$\varliminf_{n\to\infty} \left(1 + \frac{2}{n}\right) \left(1 + \cos\frac{n\pi}{6}\right)^{\dfrac{1}{n}} = 0 \quad \text{and} \quad \varlimsup_{n\to\infty} \left(1 + \frac{2}{n}\right) \left(1 + \cos\frac{n\pi}{6}\right)^{\dfrac{1}{n}} = 1.$$

Another solution. Let $a_n = \left(1 + \frac{2}{n}\right) \left(1 + \cos\frac{n\pi}{6}\right)^{\dfrac{1}{n}}$. Then

$$0 \le a_n \le \left(1 + \frac{1}{n}\right) \cdot 2^{\frac{1}{n}}$$

because $-1 \le \cos\frac{n\pi}{6} \le 1$. Thus

$$0 \le \varliminf_{n\to\infty} a_n \le \varlimsup_{n\to\infty} a_n \le \varlimsup_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot 2^{\frac{1}{n}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot 2^{\frac{1}{n}} = 1.$$

Since

$$\lim_{k\to\infty} a_{12k+6} = \lim_{k\to\infty} \left(1 + \frac{1}{12k + 6}\right) (1 + \cos(2k\pi + \pi))^{\frac{1}{12k+6}} = \lim_{k\to\infty} 0 = 0,$$

$$0 \le \varliminf_{n\to\infty} a_n \le \lim_{k\to\infty} a_{12k+6} = 0.$$

It follows that

$$\varliminf_{n\to\infty} a_n = 0.$$

Since

$$\lim_{k\to\infty} a_{12k} = \lim_{k\to\infty} \left(1 + \frac{1}{12k}\right) \cdot 2^{\frac{1}{12k}} = 1,$$

$$1 = \lim_{k\to\infty} a_{12k} \le \varlimsup_{n\to\infty} a_n \le 1.$$

It follows that

$$\varlimsup_{n\to\infty} a_n = 1.$$

□

10. Let $\sum_{n=1}^{\infty} a_n$ be a **convergent positive** series. Show that the series $\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n + a_n^2}$ is also convergent.

Proof. Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Let $b_n = \dfrac{a_n}{1+a_n+a_n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \frac{1+a_n+a_n^2}{a_n} = \lim_{n\to\infty} (1+a_n+a_n^2) = 1+0+0 = 1$$

and so the series $\sum_{n=1}^{\infty} \dfrac{a_n}{1+a_n+a_n^2}$ is convergent by the limit comparison test.

Another Solution. Since $a_n > 0$,

$$\frac{a_n}{1+a_n+a_n^2} \leq a_n.$$

By comparison test, $\sum_{n=1}^{\infty} \dfrac{a_n}{1+a_n+a_n^2}$ converges because $\sum_{n=1}^{\infty} a_n$ converges. □