NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2002-2003

MA2108 Advanced Calculus II

November 2002 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.
3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\sqrt{n^2+2n}-n\right\}$.

(b) $\left\{(6^n+8^n)^{\frac{1}{n}}\right\}$.

(c) $\left\{\left(\frac{3n}{3n-2}\right)^{2n+\sqrt{n}}\right\}$.

(d) $\left\{\frac{n^{100}\cdot 100^n\cdot\cos n}{n!}\right\}$.

Solution. (a).

$$\lim_{n\to\infty}\sqrt{n^2+2n}-n=\lim_{n\to\infty}\frac{n^2+2n-n^2}{\sqrt{n^2+2n}+n}=\lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{2}{n}}+1}=1.$$

(b).

$$\lim_{n\to\infty}(6^n+8^n)^{\frac{1}{n}}=\lim_{n\to\infty}8\left(1+\left(\frac{6}{8}\right)^n\right)^{\frac{1}{n}}=8.$$

another solution:

$$8\le(6^n+8^n)^{\frac{1}{n}}\le(2\cdot 8^n)^{\frac{1}{n}}=8\cdot 2^{\frac{1}{n}}.$$

Since $\lim\limits_{n\to\infty}8\cdot 2^{\frac{1}{n}}=8$,

$$\lim_{n\to\infty}(6^n+8^n)^{\frac{1}{n}}=8$$

by the Squeeze theorem.

(c).

$$\lim_{n\to\infty}\left(\frac{3n}{3n-2}\right)^{2n+\sqrt{n}}=\lim_{n\to\infty}\frac{1}{\left[\left(1+\frac{-2/3}{n}\right)^n\right]^{2+\frac{\sqrt{n}}{n}}}=\frac{1}{\left(e^{-\frac{2}{3}}\right)^2}=e^{\frac{4}{3}}.$$

(d).

$$-\frac{n^{100}\cdot 100^n}{n!} \le \frac{n^{100}\cdot 100^n\cdot\cos n}{n!} \le \frac{n^{100}\cdot 100^n}{n!}.$$

Since

$$\lim_{n\to\infty}\frac{n^{100}\cdot 100^n}{n!} = \lim_{n\to\infty}\frac{n^{100}}{2^n}\cdot\frac{200^n}{n!} = 0\cdot 0 = 0,$$

$$\lim_{n\to\infty}\frac{n^{100}\cdot 100^n\cdot\cos n}{n!} = 0$$

by the Squeeze theorem. ∎

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+3n-1}.$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(1+2\ln n)}.$

(c) $\displaystyle\sum_{n=1}^{\infty}5^n\left(1-\frac{2}{n+3}\right)^{n^2}.$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n}.$

Solution. (a). Let $a_n = \dfrac{\sqrt{n}}{n^2+3n-1}$ and $b_n = \dfrac{1}{n^{\frac{3}{2}}}$. Since

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{n^2+3n-1}\cdot n^{\frac{3}{2}} = 1$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges by the p-series, the series $\displaystyle\sum_{n=1}^{\infty}a_n$ is convergent by the limit comparison test.

(b). Let $f(x) = \dfrac{1}{x(1+2\ln x)}$. Then $f(x)$ is positive and monotone decreasing on $[1,+\infty)$. Since the integral

$$\int_1^{\infty}\frac{1}{x(1+2\ln x)}\,dx \xlongequal[dy=\frac{dx}{x}]{y=\ln x} \int_0^{\infty}\frac{1}{1+2y}\,dy = \frac{1}{2}\ln(y+\frac{1}{2})\Big|_0^{\infty} = +\infty$$

diverges, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(1+2\ln n)}$ diverges by the integral test.

(c). Let $a_n = 5^n \left(1 - \dfrac{2}{n+3}\right)^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} 5\left(1 - \frac{2}{n+3}\right)^n = 5 \cdot e^{-2} = \frac{5}{e^2} < 1.$$

Thus the series converges by the (simplified) root test.

(d).

$$\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \le \frac{1}{n\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}.$$

Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by the p-series, the series converges by the comparison test.

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Question 3 [10 marks]

Find the radius of convergence of each of the following power series. Justify your answer.

(a) $\displaystyle\sum_{k=1}^{\infty}\left(1-\frac{2}{k}\right)^{k^2}(x-1)^k.$

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{(k!)^2}{(2k)!}(2x+1)^k.$

Solution. (a).

$$R=\frac{1}{\overline{\lim\limits_{n\to\infty}}\sqrt[n]{\left|\left(1-\frac{2}{n}\right)^{n^2}\right|}}=\frac{1}{\lim\limits_{n\to\infty}\left(1-\frac{2}{n}\right)^{n}}=\frac{1}{e^{-2}}=e^2.$$

(b). $\displaystyle\sum_{k=1}^{\infty}\frac{(k!)^2}{(2k)!}(2x+1)^k=\sum_{k=1}^{\infty}\frac{(k!)^2\cdot 2^k}{(2k)!}\left(x+\frac{1}{2}\right)^k.$

$$R=\frac{1}{\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}=\frac{1}{\lim\limits_{n\to\infty}\frac{((n+1)!)^2\cdot 2^{n+1}\cdot(2n)!}{(2n+2)!\cdot(n!)^2\cdot 2^n}}=\frac{1}{\lim\limits_{n\to\infty}\frac{(n+1)^2\cdot 2}{(2n+2)(2n+1)}}=2.$$

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Question 4 [18 marks]

(a) Find limit inferior and limit superior of each of the following sequences.

(i) $\left\{\left[(-1)^n-\frac{1}{2}\right]^n\right\}.$

(ii) $\left\{\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{4}\right)^{\frac{1}{n}}\right\}.$

(b) Is the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2\ln n+1}{3\sqrt{n}}$ absolutely convergent, conditionally convergent or divergent? Justify your answer.

Solution. (a)(i). Let $a_n = \left[(-1)^n - \frac{1}{2}\right]^n$.

$$\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} \left(-1 - \frac{1}{2}\right)^{2n-1} = -\infty$$

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \left(1 - \frac{1}{2}\right)^{2n} = 0.$$

The subsequential limits are 0 and $-\infty$. Thus

$$\overline{\lim_{n\to\infty}}\, a_n = 0 \qquad \underline{\lim_{n\to\infty}}\, a_n = -\infty.$$

(ii). Let $a_n = \left(1 + \frac{1}{n}\right)\left(1 + \sin\frac{n\pi}{4}\right)^{\frac{1}{n}}$. Let $n = 8k + r$, where $k \geq 0$ and $r = 0, 1, 2, \cdots, 7$. Then

$$a_{8k+r} = \left(1 + \frac{1}{8k+r}\right)\left(1 + \sin(2k\pi + \frac{r\pi}{4})\right)^{\dfrac{1}{8k+r}}$$

$$= \left(1 + \frac{1}{8k+r}\right)\left(1 + \sin\frac{r\pi}{4}\right)^{\dfrac{1}{8k+r}}.$$

For $r \neq 6$,

$$1 + \sin\frac{r\pi}{4} \geq 1 - \frac{\sqrt{2}}{2}$$

and so

$$\lim_{k\to\infty} a_{8k+r} = (1 + 0)\cdot 1 = 1$$

for $r \neq 6$. For $r = 6$,

$$a_{8k+6} = \left(1 + \frac{1}{8k+6}\right)\left(1 + \sin\frac{6\pi}{4}\right)^{\dfrac{1}{8k+6}} = 0$$

and $\lim_{k\to\infty} a_{8k+6} = 0$. The subsequential limits are 0 and 1. Thus

$$\overline{\lim_{n\to\infty}}\, a_n = 1 \qquad \underline{\lim_{n\to\infty}}\, a_n = 0.$$

Another solution of a (ii). Let $a_n = \left(1 + \frac{1}{n}\right)\left(1 + \sin\frac{n\pi}{4}\right)^{\frac{1}{n}}$. Then

$$0 \leq a_n \leq \left(1 + \frac{1}{n}\right)\cdot 2^{\frac{1}{n}}$$

because $-1 \le \sin \dfrac{n\pi}{4} \le 1$. Thus

$$0 \le \varliminf_{n\to\infty} a_n \le \varlimsup_{n\to\infty} a_n \le \varlimsup_{n\to\infty} \left(1+\frac{1}{n}\right)\cdot 2^{\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\cdot 2^{\frac{1}{n}} = 1.$$

Since

$$\lim_{k\to\infty} a_{8k+6} = \lim_{k\to\infty} \left(1+\frac{1}{8k+6}\right)\left(1+\sin\frac{6\pi}{4}\right)^{\dfrac{1}{8k+6}} = 0,$$

$$0 \le \varliminf_{n\to\infty} a_n \le \lim_{k\to\infty} a_{8k+6} = 0.$$

It follows that $\varliminf_{n\to\infty} a_n = 0$. Since

$$\lim_{k\to\infty} a_{8k} = \lim_{k\to\infty} \left(1+\frac{1}{8k}\right)\cdot 1 = 1,$$

$$1 = \lim_{k\to\infty} a_{8k} \le \varlimsup_{n\to\infty} a_n \le 1$$

and so $\varlimsup_{n\to\infty} a_n = 1$.

(b). Let $f(x) = \dfrac{2\ln x + 1}{3\sqrt{x}}$. Then

$$f'(x) = \frac{2\cdot\dfrac{1}{x}\cdot\sqrt{x} - (2\ln x+1)\cdot\dfrac{1}{2\sqrt{x}}}{3x} = \frac{4-2\ln x - 1}{6x\sqrt{x}} = \frac{3-2\ln x}{6x\sqrt{x}} \le 0$$

for $\ln x \ge \dfrac{3}{2}$ or $x \ge e^{\frac{3}{2}}$. Thus the sequence $\left\{\dfrac{2\ln n+1}{3\sqrt{n}}\right\}$ is eventually monotone decreasing. Since

$$\lim_{n\to\infty} \frac{2\ln n+1}{3\sqrt{n}} = \lim_{n\to\infty} \frac{2\ln n}{3\sqrt{n}} + \frac{1}{3\sqrt{n}} = 0,$$

the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{2\ln n+1}{3\sqrt{n}}$ converges by the alternating series test.

Since

$$\left|(-1)^n\frac{2\ln n+1}{3\sqrt{n}}\right| \ge \frac{1}{3\sqrt{n}}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3\sqrt{n}} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ diverges by the p-series, the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^n\frac{2\ln n+1}{3\sqrt{n}}\right|$ diverges by the comparison test.

In conclusion, the series $\sum_{n=1}^{\infty}(-1)^n \frac{2\ln n+1}{3\sqrt{n}}$ converges conditionally.

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SECTION B

Answer not more than **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Evaluate $\lim_{n\to\infty}\int_0^{\frac{1}{2}} \frac{x^n \sin nx}{1+x^n}\, dx$. Justify your answer.

(b) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(1-2x)^n}{3n+1}$. Justify your answer.

(c) Let $\{a_n\}$ be a bounded sequence of real numbers. Show that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \leq 1.$$

Solution. (a). For $0 \leq x \leq \frac{1}{2}$,

$$0 \leq \left|\frac{x^n \sin nx}{1+x^n}\right| \leq \frac{x^n}{1+x^n} \leq \frac{\left(\frac{1}{2}\right)^n}{1}.$$

Since $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$, $\lim_{n\to\infty}\left|\frac{x^n \sin nx}{1+x^n}\right| = 0$ by the Squeeze Theorem and so

$$F(x) = \lim_{n\to\infty}\frac{x^n \sin nx}{1+x^n} = 0$$

for $0 \leq x \leq \frac{1}{2}$. Now

$$T_n = \sup_{0\leq x\leq\frac{1}{2}} |F_n(x) - F(x)| = \sup_{0\leq x\leq\frac{1}{2}} \left|\frac{x^n \sin nx}{1+x^n}\right| \leq \left(\frac{1}{2}\right)^n.$$

Since $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze Theorem and so $\{F_n(x)\}$ converges uniformly to 0 on $[0, \frac{1}{2}]$. Thus

$$\lim_{n\to\infty}\int_0^{\frac{1}{2}} \frac{x^n \sin nx}{1+x^n}\,dx = \int_0^{\frac{1}{2}} \lim_{n\to\infty} \frac{x^n \sin nx}{1+x^n}\,dx = \int_0^{\frac{1}{2}} 0\,dx = 0.$$

(b). $\sum_{n=1}^{\infty} \frac{(1-2x)^n}{3n+1} = \sum_{n=1}^{\infty} \frac{(-2)^n}{3n+1}\left(x-\frac{1}{2}\right)^n$. The radius of convergence

$$R = \frac{1}{\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim_{n\to\infty}\frac{2^{n+1}\cdot(3n+1)}{2^n\cdot(3n+4)}} = \frac{1}{2}.$$

For $x = \frac{1}{2} + \frac{1}{2} = 1$, the series

$$\sum_{n=1}^{\infty} \frac{(1-2x)^n}{3n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}$$

is convergent by the alternating series test. For $x = \frac{1}{2} - \frac{1}{2} = 0$, the series

$$\sum_{n=1}^{\infty} \frac{(1-2x)^n}{3n+1} = \sum_{n=1}^{\infty} \frac{1}{3n+1}$$

is divergent by the limit comparison test with respect to $\sum_{n=1}^{\infty}\frac{1}{n}$ because

$$\lim_{n\to\infty} \frac{\frac{1}{3n+1}}{\frac{1}{n}} = \frac{1}{3}$$

and the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. In conclusion, the interval of convergence is $(0, 1]$.

(c). Since $\{a_n\}$ is a bounded sequence, there exists non-zero numbers m and M such that

$$m \le a_n \le M$$

for all n and so

$$|a_n| \le \max\{|m|, |M|\}$$

for all n. Write c for $\max\{|m|, |M|\}$. Then

$$\overline{\lim_{n\to\infty}}\, |a_n|^{\frac{1}{n}} \le \overline{\lim_{n\to\infty}}\, c^{\frac{1}{n}} = \lim_{n\to\infty} c^{\frac{1}{n}} = 1,$$

where $\overline{\lim_{n\to\infty}} c^{\frac{1}{n}} = \lim_{n\to\infty} c^{\frac{1}{n}}$ because the limit of $c^{\frac{1}{n}}$ exists. ∎

Question 6 [20 marks]

(a) Consider the function

$$f(x) = \sum_{n=1}^{\infty} x^n e^{-nx}.$$

Is $f(x)$ continuous on $[0, +\infty)$? Justify your answer.

(b) Consider the sequence $\{x_n\}$ defined recursively by

$$x_1 = 2, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right), \quad \text{for} \quad \text{n} \geq 1.$$

Show that $\{x_n\}$ converges, and find its limit.

(c) Show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1+2x}}$ does not converge uniformly on $(0, +\infty)$.

Solution. (a). Let $f_n(x) = x^n e^{-nx}$. From

$$f_n'(x) = nx^{n-1}e^{-nx} - nx^n e^{-nx} = nx^{n-1}e^{-nx}(1-x) = 0$$

$x = 0, 1$. Since $f_n(0) = \lim_{x\to\infty} f_n(x) = 0$ and $f_n(1) = e^{-n}$, the maximum of $f_n(x)$ on $[0, +\infty)$ is $f_n(1) = e^{-n}$, that is,

$$|f_n(x)| = f_n(x) \leq e^{-n}$$

for $x \in [0, +\infty)$. Since the series $\displaystyle\sum_{n=1}^{\infty} e^{-n}$ converges, the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly by the Weierstrass M-test and hence the function

$$f(x) = \sum_{n=1}^{\infty} x^n e^{-nx}$$

is continuous.

(b). First we show that $x_n \geq \sqrt{3}$ by induction. $x_1 \geq \sqrt{3}$. Suppose that $x_n \geq \sqrt{3} > 0$. Then

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right) \geq \frac{1}{2} \cdot 2 \cdot \sqrt{x_n} \cdot \sqrt{\frac{3}{x_n}} = \sqrt{3}.$$

The induction is finished and so $x_n \geq \sqrt{3}$ for all n.

Now we show by induction that $\{x_n\}$ is monotone decreasing.

$$x_2 = \frac{1}{2}\left(2 + \frac{3}{2}\right) = 1 + \frac{3}{4} < x_1.$$

Suppose that $x_{n-1} \geq x_n$. Then

$$x_n - x_{n+1} = \frac{1}{2}\left(x_{n-1} + \frac{3}{x_{n-1}}\right) - \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$$

$$= \frac{1}{2}\left[(x_{n-1} - x_n) + \left(\frac{3}{x_{n-1}} - \frac{3}{x_n}\right)\right] = \frac{1}{2}\left[(x_{n-1} - x_n) + \frac{3(x_n - x_{n-1})}{x_{n-1}x_n}\right]$$

$$= \frac{1}{2} \cdot (x_{n-1} - x_n) \cdot \frac{x_{n-1}x_n - 3}{x_{n-1}x_n} \geq 0$$

because $x_{n-1}x_n - 3 \geq \sqrt{3} \cdot \sqrt{3} - 3 = 0$ by the first step.

By the Monotone Convergence Theorem, the sequence $\{x_n\}$ converges because it is monotone decreasing and bounded below. Let $A = \lim\limits_{n\to\infty} x_n$. Then

$$A = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{3}{x_n}\right) = \frac{1}{2}\left(A + \frac{3}{A}\right).$$

It follows that $A = \pm\sqrt{3}$. Since each $x_n \geq \sqrt{3}$, $A \neq -\sqrt{3}$ and so $A = \sqrt{3}$.

(c) On $(0, \infty)$,

$$T_n = \sup_{x>0} \left| \sum_{k=n+1}^{\infty} \frac{1}{k^{1+2x}} \right|$$

$$= \sup_{x>0} \sum_{k=n+1}^{\infty} \frac{1}{k^{1+2x}}$$

$$\geq \sup_{x>0} \left(\frac{1}{(n+1)^{1+2x}} + \frac{1}{(n+2)^{1+2x}} + \cdots + \frac{1}{(2n)^{1+2x}} \right)$$

$$\geq \sup_{x>0} \left(\frac{1}{(2n)^{1+2x}} + \frac{1}{(2n)^{1+2x}} + \cdots + \frac{1}{(2n)^{1+2x}} \right)$$

$$= \sup_{x>0} \frac{n}{(2n)^{1+2x}} = \sup_{x>0} \frac{1}{2 \cdot (2n)^{2x}}$$

Since the function $\dfrac{1}{2 \cdot (2n)^{2x}}$ is monotone decreasing on $(0, \infty)$ with $\lim\limits_{x\to 0} \dfrac{1}{2 \cdot (2n)^{2x}} = \dfrac{1}{2}$, we have $\sup\limits_{x>0} \dfrac{1}{2 \cdot (2n)^{2x}} = \dfrac{1}{2}$ and so

$$T_n \geq \frac{1}{2}.$$

It follows that T_n does not tend to 0 as n tends to ∞, and so the series of functions $\sum_{k=1}^{\infty} \frac{1}{k^{1+2x}}$ does not converge uniformly on $(0, \infty)$.

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Question 7 [20 marks]

(a) Let $f(x) = x^5 \sin(x^9)$. Find $f^{(48)}(0)$.

(b) Let A and B be two non-empty bounded set of real numbers. Define $A + B = \{a + b \mid a \in A,\ b \in B\}$. Prove that

$$\inf A + \inf B = \inf(A + B).$$

(c) Show that the series of functions $\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)^x}$ converges uniformly on $[1, +\infty)$.

Solution. (a). From the Taylor series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$f(x) = x^5 \sin\left(x^9\right) = x^{14} - \frac{x^{32}}{3!} + \frac{x^{50}}{5!} - \frac{x^{68}}{7!} + \cdots.$$

Thus

$$\frac{f^{(48)}(0)}{48!} = 0$$

and so $f^{(48)}(0) = 0$.

(b). For each $a \in A$ and $b \in B$, $\inf A \le a$ and $\inf B \le b$. Thus

$$\inf A + \inf B \le a + \inf B \le a + b$$

for any $a \in A$ and $b \in B$. Hence $\inf A + \inf B$ is a lower bound of the set $A + B$ and so

$$\inf A + \inf B \le \inf(A + B)$$

because $\inf(A + B)$ is the greatest lower bound of $A + B$.

Given any $a \in A$, for any $b \in B$,

$$\inf(A + B) \le a + b \qquad \Rightarrow \qquad \inf(A + B) - a \le b.$$

Thus $\inf(A + B) - a$ is a lower bound of B for any given a and so

$$\inf(A + B) - a \le \inf B \qquad \Rightarrow \inf(A + B) - \inf B \le a$$

for any given $a \in A$ because $\inf B$ is the greatest lower bound of B. This shows that $\inf(A+B) - \inf B$ is a lower bound of A and so

$$\inf(A+B) - \inf B \le \inf A, \qquad \text{that is,} \qquad \inf(A+B) \le \inf A + \inf B$$

because $\inf A$ is the greatest lower bound of A.

Hence $\inf A + \inf B = \inf(A+B)$.

(c). For each $x \in [1, \infty)$, by the alternating series estimation,

$$\left| \sum_{k=n+1}^{\infty} (-1)^k \frac{1}{(2k+1)^x} \right| \le \frac{1}{(2n+3)^x}$$

because the sequence $\left\{ \dfrac{1}{(2n+1)^x} \right\}$ is monotone decreasing and

$$\lim_{n\to\infty} \frac{1}{(2n+1)^x} = 0.$$

Thus on $[1, +\infty)$

$$\begin{aligned} 0 \le T_n &= \sup_{x\ge 1} \left| \sum_{k=n+1}^{\infty} (-1)^k \frac{1}{(2k+1)^x} \right| \\ &\le \sup_{x\ge 1} \frac{1}{(2n+3)^x} = \frac{1}{2n+3}. \end{aligned}$$

Since $\lim\limits_{n\to\infty} \dfrac{1}{2n+3} = 0$, $\lim\limits_{n\to\infty} T_n = 0$ by the Squeeze theorem. Thus the series of functions $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)^x}$ converges uniformly on $[1, +\infty)$. ∎