Solution to Take-home Exam 1

Question 1. [20 marks] Prove the following limits by using $\epsilon - N$ definition

i) $\lim_{n\to\infty} \frac{3n+8}{2n+9} = \frac{3}{2}$.

ii) $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0$.

Proof. (i). Note that

$$\left|\frac{3n+8}{2n+9} - \frac{3}{2}\right| = \left|\frac{2(3n+8)-3(2n+9)}{2(2n+9)}\right| = \frac{11}{2(2n+9)} < \frac{11}{4n} < \frac{3}{n}.$$

Given $\epsilon > 0$, choose N such that $\frac{3}{N} \le \epsilon \iff N \ge \frac{3}{\epsilon}$. When $n > N$,

$$\left|\frac{3n+8}{2n+9} - \frac{3}{2}\right| < \frac{3}{n} < \frac{3}{N} \le \epsilon.$$

(ii). Note that

$$\left|\frac{(-1)^n n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}.$$

Given $\epsilon > 0$, choose N such that $\frac{1}{N} \le \epsilon \iff N \ge \frac{1}{\epsilon}$. When $n > N$,

$$\left|\frac{(-1)^n n}{n^2+1} - 0\right| < \frac{1}{n} < \frac{1}{N} \le \epsilon.$$

□

Question 2. [40 marks] For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\left(\sqrt{n^2+n} - n\right)\right\}$.

(b) $\left\{(2^n + 3^n)^{\frac{1}{n}}\right\}$.

(c) $\left\{\sqrt[4]{\frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n}}\right\}$.

(d) $\left\{\left(\frac{3n}{3n-1}\right)^{2n+\sqrt{n}}\right\}$.

(e) $\left\{\frac{n^{50} \cdot 50^n \cdot \sin n}{n!}\right\}$.

Solution. (a).

$$\lim_{n\to\infty}\left(\sqrt{n^2+n} - n\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n^2+n} - n\right)\cdot\left(\sqrt{n^2+n} + n\right)}{\sqrt{n^2+n}+n}$$

$$= \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1} = \frac{1}{2}.$$

(b).

$$\lim_{n\to\infty}(2^n+3^n)^{\frac{1}{n}}=\lim_{n\to\infty}3\left[\left(\frac{2}{3}\right)^n+1\right]^{\frac{1}{n}}=3\cdot(0+1)^0=3.$$

Another solution: Since

$$3=(3^n)^{\frac{1}{n}}\le(2^n+3^n)^{\frac{1}{n}}\le(3^n+3^n)^{\frac{1}{n}}=2^{\frac{1}{n}}\cdot 3\quad\text{and}\quad\lim_{n\to\infty}2^{\frac{1}{n}}\cdot 3=3,$$

$$\lim_{n\to\infty}(2^n+3^n)^{\frac{1}{n}}=3\qquad\text{by the Squeeze Thoerem.}$$

(c).

$$\lim_{n\to\infty}\sqrt[4]{\frac{n!+2n^5+\ln n}{n!+5^n+3n}}=\lim_{n\to\infty}\sqrt[4]{\frac{1+2\frac{n^5}{n!}+\frac{\ln n}{n!}}{1+\frac{5^n}{n!}+3\frac{n}{n!}}}=\sqrt[4]{\frac{1+0+0}{1+0+0}}=1.$$

(d).

$$\lim_{n\to\infty}\left(\frac{3n}{3n-1}\right)^{2n+\sqrt{n}}=\lim_{n\to\infty}\frac{1}{\left(\frac{3n-1}{3n}\right)^{2n+\sqrt{n}}}=\lim_{n\to\infty}\frac{1}{\left[\left(1+\frac{-1}{3n}\right)^{3n}\right]^{\frac{2n+\sqrt{n}}{3n}}}$$

$$=\lim_{n\to\infty}\frac{1}{\left[\left(1+\frac{-1}{3n}\right)^{3n}\right]^{\frac{2+\frac{1}{\sqrt{n}}}{3}}}=\frac{1}{(e^{-1})^{\frac{2}{3}}}=e^{\frac{2}{3}}.$$

(e). Note that

$$-\frac{n^{50}\cdot 50^n}{n!}\le\frac{n^{50}\cdot 50^n\cdot\sin n}{n!}\le\frac{n^{50}\cdot 50^n}{n!}.$$

Since

$$\lim_{n\to\infty}\frac{n^{50}\cdot 50^n}{n!}=\lim_{n\to\infty}\frac{n^{50}}{2^n}\cdot\frac{2^n\cdot 50^n}{n!}=\lim_{n\to\infty}\frac{n^{50}}{2^n}\frac{100^n}{n!}=0\cdot 0=0,$$

$$\lim_{n\to\infty}\frac{n^{50}\cdot 50^n\cdot\sin n}{n!}=0\qquad\text{by the Squeeze Theorem.}$$

□

Question 3. [20 marks] Let f and g be real-valued function defined on a nonempty set E. Prove each of the following.

(a) If $f(x)\le g(x)$ for all $x\in E$, then $\inf\{f(x)\mid x\in E\}\le\inf\{g(x)\mid x\in E\}$.
(b) $\inf\{f(x)\mid x\in E\}+\inf\{g(x)\mid x\in E\}\le\inf\{f(x)+g(x)\mid x\in E\}$.

Proof. (a). Since

$$\inf\{f(x)\mid x\in E\}\le f(x)\le g(x)\qquad\text{for any}\quad x\in E,$$

we have $\inf\{f(x)\mid x\in E\}$ is a lower bound of the set $\{g(x)\mid x\in E\}$. By definition, $\inf\{g(x)\mid x\in E\}$ is the greatest lower bound of the set $\{g(x)\mid x\in E\}$, we have

$$\inf\{f(x)\mid x\in E\}\le\inf\{g(x)\mid x\in E\}.$$

(b). Since $\inf\{f(x) \mid x \in E\} \le f(x)$ and $\inf\{g(x) \mid x \in E\} \le g(x)$ for any $x \in E$, we have

$$\inf\{f(x) \mid x \in E\} + \inf\{g(x) \mid x \in E\} \le f(x) + g(x) \qquad \text{for any} \quad x \in E.$$

It follows that $\inf\{f(x) \mid x \in E\} + \inf\{g(x) \mid x \in E\}$ is a lower bound of the set $\{f(x) + g(x) \mid x \in E\}$. Thus

$$\inf\{f(x) \mid x \in E\} + \inf\{g(x) \mid x \in E\} \le \inf\{f(x) + g(x) \mid x \in E\}$$

because $\inf\{f(x) + g(x) \mid x \in E\}$ is the greatest lower bound of the set $\{f(x) + g(x) \mid x \in E\}$. □

Question 4. [20 marks] For A and B, subsets of $\mathbb{R}$, define $A + B = \{a + b \mid a \in A,\ b \in B\}$. Prove that

$$\sup(A + B) = \sup A + \sup B.$$

Proof. Let $x = a + b$, $a \in A$ and $b \in B$, be any point in $A + B$. Then

$$x = a + b \le \sup A + \sup B \qquad \text{because} \quad a \le \sup A \quad \text{and} \quad b \le \sup B.$$

Thus $\sup A + \sup B$ is an upper bound of the set $A + B$, and so

$$\sup(A + B) \le \sup A + \sup B \qquad \text{because superior is the least upper bound.}$$

Let a be any fixed point in A and let b be any point in B. Then $a + b \in A + B$ be by the definition, and so

$$a + b \le \sup(A + B) \qquad \text{because superior is the (least) upper bound}$$
$$\Longrightarrow b \le \sup(A + B) - a \qquad \text{for any} \quad b \in B.$$

Thus $\sup(A + B) - a$ is an upper bound of B, and so

$$\sup B \le \sup(A + B) - a \qquad \text{because} \quad \sup B \quad \text{the least upper bound of} \quad B$$
$$\Longrightarrow a \le \sup(A + B) - \sup B \qquad \text{for any} \quad a \in A.$$

Hence

$$\sup A \le \sup(A + B) - \sup B \qquad \text{because} \quad \sup A \quad \text{the least upper bound of} \quad A$$
$$\Longrightarrow \sup A + \sup B \le \sup(A + B).$$

Together with $\sup(A + B) \le \sup A + \sup B$, we have

$$\sup A + \sup B = \sup(A + B).$$

□