2002/2003 Semester I MA2108 Advanced Calculus II

Take-home Exam 2

Question 1. [40 marks] Find limit inferior and limit superior of each of the following sequences.

(a) $\left\{(1+(-1)^n)\sin\frac{n\pi}{4}\right\}$.

(b) $\left\{\frac{n+(-1)^n n^2}{n^2+1}\right\}$.

(c) $\{[1.5+(-1)^n]^n\}$.

(d) $\left\{\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}}\right\}$

Solution. (a).

$$(1+(-1)^n)\sin\frac{n\pi}{4}=\begin{cases} 0 & n=8k+1,8k+3,8k+5,8k+7\\[2ex] 0 & n=8k,8k+4\\[2ex] 2\sin\frac{2\pi}{4}=2 & n=8k+2\\[2ex] 2\sin\frac{6\pi}{4}=2\sin\frac{3\pi}{2}=-2 & n=8k+6.\end{cases}$$

Thus the set of subsequential limits are $\{-2,0,2\}$ and so

$$\underline{\lim}\,(1+(-1)^n)\sin\frac{n\pi}{4}=-2 \qquad\text{and}\qquad \overline{\lim}\,(1+(-1)^n)\sin\frac{n\pi}{4}=2.$$

(b).

$$\frac{n+(-1)^n n^2}{n^2+1}=\frac{\frac{1}{n}+(-1)^n}{1+\frac{1}{n^2}}=\begin{cases} =\dfrac{\frac{1}{2k}+1}{1+\frac{1}{(2k)^2}} & n=2k\\[4ex] =\dfrac{\frac{1}{2k+1}-1}{1+\frac{1}{(2k+1)^2}} & n=2k+1.\end{cases}$$

Since

$$\lim_{k\to\infty}\frac{\frac{1}{2k}+1}{1+\frac{1}{(2k)^2}}=\frac{1}{1}=1 \qquad\text{and}\qquad \lim_{k\to\infty}\frac{\frac{1}{2k+1}-1}{1+\frac{1}{(2k+1)^2}}=\frac{-1}{1}=-1,$$

the set of the subsequential limits is $\{-1,1\}$, and so

$$\overline{\lim}\,\frac{n+(-1)^n n^2}{n^2+1}=1 \qquad \text{and} \quad \underline{\lim}\,\frac{n+(-1)^n n^2}{n^2+1}=-1.$$

(c).

$$[1.5+(-1)^n]^n=\begin{cases} 2.5^{2k} & n=2k \\ \\ 0.5^{2k+1} & n=2k+1. \end{cases}$$

Since $\lim\limits_{k\to\infty} 2.5^{2k}=+\infty$ and $\lim\limits_{k\to\infty} 0.5^{2k+1}=0$,

$$\overline{\lim}\,[1.5+(-1)^n]^n=+\infty \qquad \text{and} \qquad \underline{\lim}\,[1.5+(-1)^n]^n=0.$$

(d). Let $n=16k+r$, where k and r are integers with $r=0,1,\cdots,15$ and $k\geq 0$. Then

$$1+\sin\frac{n\pi}{8}=1+\sin(2k\pi+\frac{r\pi}{8})=1+\sin\frac{r\pi}{8}.$$

If $r\neq 12$, then $1+\sin\dfrac{r\pi}{8}\geq 1-\dfrac{\sqrt{2}}{2}$, and so, for $r\neq 12$,

$$\left(1+\frac{1}{16k+r}\right)\cdot\left(1-\frac{\sqrt{2}}{2}\right)^{\dfrac{1}{16k+r}}$$

$$\leq\left(1+\frac{1}{16k+r}\right)\left(1+\sin\frac{(16k+r)\pi}{8}\right)^{\dfrac{1}{16k+r}}$$

$$\leq\left(1+\frac{1}{16k+r}\right)\cdot 2^{\frac{1}{16k+r}}.$$

Since

$$\lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)\cdot\left(1-\frac{\sqrt{2}}{2}\right)^{\frac{1}{16k+r}}=1\cdot\left(1-\frac{\sqrt{2}}{2}\right)^0=1$$

$$\lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)\cdot 2^{\frac{1}{16k+r}}=1\cdot 2^0=1,$$

$$\lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)\left(1+\sin\frac{(16k+r)\pi}{8}\right)^{\dfrac{1}{16k+r}}=1$$

for $r\neq 12$ by the Squeeze Theorem. When $r=12$,

$$\lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)\left(1+\sin\frac{(16k+r)\pi}{8}\right)^{\dfrac{1}{16k+r}}$$

$$=\lim_{k\to\infty}\left(1+\frac{1}{16k+r}\right)(1-1)^{\dfrac{1}{16k+r}}=\lim_{k\to\infty}0=0.$$

Thus the set of subsequential limits is $\{0, 1\}$ and so

$$\underline{\lim}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}} = 0 \quad \text{and} \quad \overline{\lim}\left(1+\frac{1}{n}\right)\left(1+\sin\frac{n\pi}{8}\right)^{\frac{1}{n}} = 1.$$

□

Question 2. [40 marks] Let $\alpha > 0$. Choose $x_1 \geq \sqrt{\alpha}$. For $n = 1, 2, 3, \cdots$, define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) Show that the sequence $\{x_n\}$ is bounded below by $\sqrt{\alpha}$ and monotone decreasing.

(b) Prove that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

(c) Prove that $0 \leq x_n - \sqrt{\alpha} \leq \frac{x_n^2 - \alpha}{x_n}$.

(d) Let $\alpha = 3$ and $x_1 = 2$. Use part (c) to find x_n such that $\left|x_n - \sqrt{3}\right| < 10^{-8}$.

Solution. (a). We show by induction that $x_n \geq \sqrt{\alpha}$ for each n. By the assumption, $x_1 \geq \sqrt{\alpha}$. Suppose that $x_n \geq \sqrt{\alpha} > 0$. Then

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \geq \frac{1}{2}\cdot 2\sqrt{x_n}\cdot\sqrt{\frac{\alpha}{x_n}} = \sqrt{\alpha}.$$

The induction is finished and so $x_n \geq \sqrt{\alpha}$ for all n.

Now

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - x_n = \frac{1}{2}\left(\frac{\alpha}{x_n} - x_n\right)$$

$$\frac{\alpha - x_n^2}{2x_n} \leq 0 \qquad \text{because} \quad x_n \geq \sqrt{\alpha}.$$

Thus $\{x_n\}$ is monotone decreasing.

(b). Let $A = \lim_{n\to\infty} x_n$. Then

$$A = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{1}{2}\left(A + \frac{\alpha}{A}\right)$$

$$\Longrightarrow 2A = A + \frac{\alpha}{A}$$

$$\Longrightarrow A = \sqrt{\alpha} \qquad \text{because} A \geq \sqrt{\alpha} > 0.$$

(c). Since $x_n \geq \sqrt{\alpha}$, we have $0 \leq x_n - \sqrt{\alpha}$. Now

$$\frac{x_n^2 - \alpha}{x_n} - (x_n - \sqrt{\alpha}) = -\frac{\alpha}{x_n} + \sqrt{\alpha}$$

$$= \frac{\sqrt{\alpha}x_n - \alpha}{x_n} \geq \frac{\sqrt{\alpha}\sqrt{\alpha} - \alpha}{x_n} = 0.$$

Thus

$$0 \leq x_n - \sqrt{\alpha} \leq \frac{x_n^2 - \alpha}{x_n}.$$

(d). We compute first few x_n using the formula $x_{n+1} = \dfrac{x_n^2+\alpha}{2x_n} = \dfrac{x_n^2+3}{2x_n}$.

$$0 \le x_1 - \sqrt{3} \le \frac{x_1^2-3}{x_1} = \frac{4-3}{2} = \frac{1}{2}.$$

$$x_2 = \frac{x_1^2+3}{2x_1} = \frac{7}{4} \qquad \frac{x_2^2-\alpha}{x_2} = \frac{\frac{49}{16}-3}{\frac{7}{4}} = \frac{1}{4\cdot 7} = \frac{1}{28}.$$

$$x_3 = \frac{x_2^2+3}{2x_2} = \frac{\frac{49}{16}+3}{2\cdot\frac{7}{4}} = \frac{97}{56} \qquad \frac{x_3^2-\alpha}{x_3} = \frac{\frac{97^2}{56^2}-3}{\frac{97}{56}} = \frac{97^2-3\cdot 56^2}{56\cdot 97} = \frac{1}{56\cdot 97}$$

$$x_4 = \frac{x_3^2+3}{2x_3} = \frac{\frac{97^2}{56^2}+3}{2\cdot\frac{97}{56}} = \frac{97^2+3\cdot 56^2}{2\cdot 56\cdot 97} = \frac{18817}{10864}$$

$$\frac{x_4^2-\alpha}{x_4} = \frac{\frac{18817^2}{10864^2}-3}{\frac{18817}{10864}} = \frac{18817^2-3\cdot 10864^2}{10864\cdot 18817} = \frac{354079489-354079488}{10864\cdot 18817}$$

$$= \frac{1}{10864\cdot 18817} = \frac{1}{204427888} < 10^{-8}.$$

Thus

$$\sqrt{3} \approx x_4 = \frac{18817}{10864} \qquad \text{with error} < 10^{-8}.$$

□

Question 3. [20 marks] Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$. Prove that

$$\underline{\lim}\, a_n + \overline{\lim}\, b_n \le \overline{\lim}(a_n+b_n) \le \overline{\lim}\, a_n + \overline{\lim}\, b_n.$$

Proof. First we prove that $\overline{\lim}(a_n+b_n) \le \overline{\lim}\, a_n + \overline{\lim}\, b_n$. For each $m \ge n$, we have

$$a_m + b_m \le \sup\{a_k \mid k \ge n\} + \sup\{b_k \mid k \ge n\}.$$

Thus $\sup\{a_k \mid k \ge n\} + \sup\{b_k \mid k \ge n\}$ is an upper bound of the set $\{a_m + b_m \mid m \ge n\} = \{a_k + b_k \mid k \ge n\}$. It follows that

$$\sup\{a_k + b_k \mid k \ge n\} \le \sup\{a_k \mid k \ge n\} + \sup\{b_k \mid k \ge n\}$$

because sup is the least upper bound, and so, by letting n tend to infinity,

$$\overline{\lim}(a_n+b_n) = \lim_{n\to\infty}(\sup\{a_k+b_k \mid k \ge n\}) \le \lim_{n\to\infty}(\sup\{a_k \mid k \ge n\} + \sup\{b_k \mid k \ge n\})$$

$$= \lim_{n\to\infty}(\sup\{a_k \mid k \ge n\}) + \lim_{n\to\infty}(\sup\{b_k \mid k \ge n\}) = \overline{\lim}\, a_n + \overline{\lim}\, b_n.$$

Next we show that $\underline{\lim}\, a_n + \overline{\lim}\, b_n \le \overline{\lim}(a_n+b_n)$. For each $m \ge n$,

$$\inf\{a_k \mid k \ge n\} + b_m \le a_m + b_m \le \sup\{a_k + b_k \mid k \ge n\}$$

because $\inf\{a_k \mid k \ge n\} \le a_m$ for each $m \ge n$. Thus

$$b_m \le \sup\{a_k + b_k \mid k \ge n\} - \inf\{a_k \mid k \ge n\} \qquad \text{for each} \quad m \ge n.$$

It follows that $\sup\{a_k + b_k \mid k \ge n\} - \inf\{a_k \mid k \ge n\}$ is an upper bound of the set $\{b_m \mid m \ge n\} = \{b_k \mid k \ge n\}$, and so

$$\sup\{b_k \mid k \ge n\} \le \sup\{a_k + b_k \mid k \ge n\} - \inf\{a_k \mid k \ge n\}$$

because sup is the least upper bound. By letting n tend to infinity,

$$\overline{\lim}\, b_n = \lim_{n\to\infty} (\sup\{b_k \mid k \geq n\}) \leq \lim_{n\to\infty} (\sup\{a_k + b_k \mid k \geq n\} - \inf\{a_k \mid k \geq n\})$$

$$= \lim_{n\to\infty} (\sup\{a_k + b_k \mid k \geq n\}) - \lim_{n\to\infty} (\inf\{a_k \mid k \geq n\}) = \overline{\lim}(a_n + b_n) - \underline{\lim}\, a_n.$$

Thus

$$\underline{\lim}\, a_n + \overline{\lim}\, b_n \leq \overline{\lim}(a_n + b_n).$$

□