Solution to Take-home Exam 3

Question 1 [40 marks]
Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+2k-1}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(2+\ln n)}$.

(c) $\displaystyle\sum_{n=1}^{\infty} 6^n \left(1-\frac{2}{n+1}\right)^{n^2}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}$.

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{k}$.

Solution. (a). Since

$$\lim_{k\to\infty} \frac{\frac{\sqrt{k}}{k^2+2k-1}}{\frac{1}{k^{\frac{3}{2}}}} = \lim_{k\to\infty} \frac{1}{1+2/k-1/k^2} = 1,$$

and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}}$ converges by the p-series, the series converges by the limit comparison test.

(b). Let $f(x) = \dfrac{1}{x(2+\ln x)}$. Then $f(x)$ is positive and monotone decreasing on $[1,+\infty]$. Since

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x(2+\ln x)}dx$$
$$= \int_2^{\infty} \frac{dy}{y} = \ln y\Big|_2^{\infty} = +\infty$$

diverges, the series $\sum_{n=1}^{\infty} \frac{1}{n(2+\ln n)}$ diverges by the integral test.

(c). Let $a_n = 6^n \left(1-\dfrac{2}{n+1}\right)^{n^2}$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} 6\left(1-\frac{2}{n+1}\right)^n = \lim_{n\to\infty} 6\frac{\left(1-\frac{2}{n+1}\right)^{n+1}}{1-\frac{2}{n+1}} = 6e^{-2} = \frac{6}{e^2} < 1$$

and so the series is convergent by the root test.

(d). Let $a_n = \dfrac{n^n}{3^n \cdot n!}$. Since

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1} \cdot 3^n \cdot n!}{3^{n+1} \cdot (n+1)! \cdot n^n}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n \cdot (n+1)}{3 \cdot (n+1)} = \frac{e}{3} < 1,$$

the series is convergent by the ratio test.

(e). Let

$$a_k = \frac{\sqrt{k+1}-\sqrt{k}}{k} = \frac{(\sqrt{k+1}-\sqrt{k})(\sqrt{k+1}+\sqrt{k})}{k(\sqrt{k+1}+\sqrt{k})} = \frac{1}{k(\sqrt{k+1}+\sqrt{k})}$$

and let $b_k = \dfrac{1}{k^{\frac{3}{2}}}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\dfrac{1}{k(\sqrt{k+1}+\sqrt{k})}}{\dfrac{1}{k^{\frac{3}{2}}}} = \lim_{k\to\infty} \frac{k^{\frac{3}{2}}}{k(\sqrt{k+1}+\sqrt{k})} = \lim_{k\to\infty} \frac{1}{\sqrt{1+1/k}+1} = \frac{1}{2}.$$

Since $\sum_{k=1}^{\infty} b_k$ converges by the p-series, the series converges by the limit comparison test. □

Question 2. [20 marks]

(a) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, prove that $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

(b) If a and b are positive real numbers, prove that

$$\sum_{k=1}^{\infty} \frac{1}{(ak+b)^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

Proof. (a). Suppose that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges. Since $\sum_{n=1}^{\infty} a_n$ converges by the assumption,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}[(a_n + b_n) - a_n]$$

converges, which contradicts to the assumption that $\sum_{n=1}^{\infty} b_n$ diverges. Hence $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

(b). Let $a_k = \dfrac{1}{(ak+b)^p}$ and let $b_k = \dfrac{1}{k^p}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\dfrac{1}{(ak+b)^p}}{\dfrac{1}{k^p}} = \lim_{k\to\infty} \frac{k^p}{(ak+b)^p} = \lim_{k\to\infty} \frac{1}{(a+b/k)^p} = \frac{1}{a} \neq 0, \infty.$$

By the p-series, $\sum_{k=1}^{\infty} b_k$ converges if $p > 1$ and diverges if $p \leq 1$. Thus, by the limit comparison test, a_k converges if $p > 1$ and diverges if $p \leq 1$. □

Question 3. [20 marks] Determine all values of p and q for which the following series converges:

$$\sum_{k=2}^{\infty} \frac{1}{k^p (\ln k)^q}.$$

Solution. **Case I.** $p > 1$. Choose a small positive number $a > 0$ such that $p - a > 1$. Let $a_k = \dfrac{1}{k^p(\ln k)^q}$ and let $b_k = \dfrac{1}{k^{p-a}}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\dfrac{1}{k^p(\ln k)^q}}{\dfrac{1}{k^{p-a}}} = \lim_{k\to\infty} \frac{(\ln k)^{-q}}{k^a} = 0,$$

that is, $a_k << b_k$, for any $q \in \mathbb{R}$ by the standard limits. Since $p - a > 1$, the series $\sum_{k=2}^{\infty} \frac{1}{k^{p-a}}$ converges by the p-series and so the series $\sum_{k=2}^{\infty} \frac{1}{k^p(\ln k)^q}$ converges by the limit comparison test.

Case II. $p = 1$. Let $f(x) = \dfrac{1}{x(\ln x)^q}$. Then

$$f'(x) = -\frac{(\ln x)^q + x \cdot q(\ln x)^{q-1} \cdot \dfrac{1}{x}}{x^2(\ln x)^{2q}} = -\frac{\ln x + q}{x^2(\ln x)^{q+1}} < 0$$

when $\ln x + q > 0$ and $x > 1$, or $x > \max\{e^{-q}, 1\}$. Thus $f(x)$ is positive and **eventually** monotone decreasing. Now

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x(\ln x)^q}\,dx \xlongequal[dy = \frac{1}{x}dx]{y = \ln x} \int_{\ln 2}^{\infty} \frac{1}{y^q}\,dy \begin{cases} \text{converges} & q > 1 \\ \text{diverges} & q \leq 1. \end{cases}$$

Thus the series $\sum_{k=2}^{\infty} \dfrac{1}{k(\ln k)^q}$ converges if $q > 1$ and diverges if $q \leq 1$.

Case III. $p < 1$. Choose a small positive number a such that $p + a < 1$. Let $a_k = \dfrac{1}{k^p(\ln k)^q}$ and let $b_k = \dfrac{1}{k^{p+a}}$. Then

$$\lim_{k\to\infty} \frac{b_k}{a_k} = \lim_{k\to\infty} \frac{\dfrac{1}{k^{p+a}}}{\dfrac{1}{k^p(\ln k)^q}} = \lim_{k\to\infty} \frac{(\ln k)^q}{k^a} = 0,$$

that is, $b_k << a_k$, for any $q \in \mathbb{R}$ by the standard limits. Since $p + a < 1$, the series $\sum_{k=2}^{\infty} \frac{1}{k^{p+a}}$ diverges by the p-series and so the series $\sum_{k=2}^{\infty} \frac{1}{k^p(\ln k)^q}$ diverges by the limit comparison test.

In conclusion, the series

$$\sum_{k=2}^{\infty} \frac{1}{k^p(\ln k)^q} \begin{cases} \text{converges} & p > 1, q \in \mathbb{R}, \\ \text{converges} & p = 1, q > 1, \\ \text{diverges} & \text{otherwise.} \end{cases}$$

□

Question 4. [20 marks] Suppose that $\sum_{k=1}^{\infty} a_k$ is a **convergent positive** series. For each of the following, either prove that the given series converges, or provide an example for which the series diverges.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{a_k}{1 + a_k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \sqrt{a_k}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \sqrt[k]{k}\, a_k$.

(d) $\displaystyle\sum_{k=1}^{\infty} \sqrt{\frac{a_k}{k}}$.

Solution. (a). Yes. Since $\sum_{k=1}^{\infty} a_k$ converges, $\lim_{k\to\infty} a_k = 0$. Let $b_k = \dfrac{a_k}{1 + a_k}$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{a_k}{\dfrac{a_k}{1 + a_k}} = \lim_{k\to\infty} (1 + a_k) = 1.$$

By the limit comparison test, the series $\sum_{k=1}^{\infty} \frac{a_k}{1 + a_k}$ converges.

(b). No. Example: Let $a_k = \dfrac{1}{k^2}$. Then $\displaystyle\sum_{k=1}^{\infty} a_k$ is a positive convergent series, but $\displaystyle\sum_{k=1}^{\infty} \sqrt{a_k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the p-series.

(c). Yes. Let $b_k = \sqrt[k]{k} a_k$. Then

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{a_k}{\sqrt[k]{k} a_k} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} = 1.$$

By the limit comparison test, the series $\displaystyle\sum_{k=1}^{\infty} \sqrt[k]{k} a_k$ converges.

(d). No. Example: Let $a_k = \dfrac{1}{k(1+\ln k)^2}$. Then $\displaystyle\sum_{k=1}^{\infty} a_k$ is a positive convergent series by the integral test because

$$\int_1^{\infty} \frac{1}{x(1+\ln x)^2}\, dx \xlongequal[dy = \frac{1}{x}dx]{y=1+\ln x} \int_1^{\infty} \frac{1}{y^2}\, dy$$

converges. Now

$$\sum_{k=1}^{\infty} \sqrt{\frac{a_k}{k}} = \sum_{k=1}^{\infty} \sqrt{\frac{1}{k^2(1+\ln k)^2}} = \sum_{k=1}^{\infty} \frac{1}{k(1+\ln k)}$$

diverges by the integral test because

$$\int_1^{\infty} \frac{1}{x(1+\ln x)}\, dx \xlongequal[dy = \frac{1}{x}dx]{y=1+\ln x} \int_1^{\infty} \frac{1}{y}\, dy$$

diverges. □