2002/2003 Semester I MA2108 Advanced Calculus II

Solution to Take-home Exam 4

Question 1 [40 marks]
Determine the absolute convergence, conditional convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}+1}$.

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k}}{2k^2+(-1)^k}$.

(c) $\displaystyle\sum_{k=1}^{\infty}\frac{\sin kt}{k^2+3}, \qquad t\in\mathbb{R}$.

(d) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k}\ln(\ln k)}{\sqrt{\ln k}+1}$.

(e) $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k}k^k}{(k+1)^k}$.

Solution. (a). Conditional convergence. Let $a_k=\dfrac{1}{\sqrt{k}+1}$. Then $\{a_k\}$ is positive, monotone decreasing and $\lim_{k\to\infty}a_k=0$. By the alternating series test, $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\sqrt{k}+1}$ converges. Since $\dfrac{1}{\sqrt{k}+1}\ge\dfrac{1}{\sqrt{k}+\sqrt{k}}=\dfrac{1}{2\sqrt{k}}$ and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2\sqrt{k}}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$ diverges by the p-series, the series $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{\sqrt{k}+1}\right|=\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}+1}$ diverges.

(b). Absolute convergence. Since

$$\left|\frac{(-1)^k}{2k^2+(-1)^k}\right|=\frac{1}{2k^2+(-1)^k}\le\frac{1}{k^2}$$

and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by the p-series, the series $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^k}{2k^2+(-1)^k}\right|$ converges by the comparison test.

(c). Absolute convergence. Since

$$\left|\frac{\sin kt}{k^2+3}\right|\le\frac{1}{k^2+3}\le\frac{1}{k^2}$$

and $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by the p-series, the series $\displaystyle\sum_{k=1}^{\infty}\left|\frac{\sin kt}{k^2+3}\right|$ converges by the comparison test.

(d). Conditional convergence. Let $a_k = \dfrac{\ln(\ln k)}{\sqrt{\ln k}+1}$. Then $a_k \geq 0$ for all k. To see $\{a_k\}$ *eventually* monotone decreasing, let $f(x) = \dfrac{\ln(\ln x)}{\sqrt{\ln x}+1}$. Then

$$f'(x) = \frac{\dfrac{1}{\ln x}\cdot\dfrac{1}{x}(\sqrt{\ln x}+1) - \ln(\ln x)\cdot\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{\ln x}}\cdot\dfrac{1}{x}}{(\sqrt{\ln x}+1)^2}$$

$$= \frac{2(\sqrt{\ln x}+1) - \ln(\ln x)\cdot\sqrt{\ln x}}{2x\ln x(\sqrt{\ln x}+1)^2} = \frac{2\left(1+\dfrac{1}{\sqrt{\ln x}}\right) - \ln(\ln x)}{2x(\ln x)^{\frac{3}{2}}(\sqrt{\ln x}+1)^2}$$

is **eventually** negative because the denominator is positive when $x > 1$, and the numerator is negative when x is large, say when $x > e^{e^4}$,

$$2\left(1+\frac{1}{\sqrt{\ln x}}\right) - \ln(\ln x) < 2(1+1) - \ln(\ln x) = 4 - \ln(\ln x) < 0.$$

Thus $\{a_k\}$ is **eventually** monotone decreasing. Now

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{\ln(\ln k)}{\sqrt{\ln k}+1} \xlongequal[x=\ln k]{} \lim_{x\to\infty}\frac{\ln x}{\sqrt{x}+1} = \lim_{x\to\infty}\frac{\dfrac{\ln x}{x^{\frac{1}{2}}}}{1+\frac{1}{\sqrt{x}}} = \frac{0}{1} = 0.$$

By the alternating series test, the series $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k\ln(\ln k)}{\sqrt{\ln k}+1}$ converges. Since

$$\left|\frac{(-1)^k\ln(\ln k)}{\sqrt{\ln k}+1}\right| = \frac{\ln(\ln k)}{\sqrt{\ln k}+1} \geq \frac{1}{\sqrt{\ln k}+1} \geq \frac{1}{\sqrt{\ln k}+\sqrt{\ln k}} = \frac{1}{2\sqrt{\ln k}} \geq \frac{1}{2\sqrt{k}}$$

for $\ln(\ln k) \geq 1$ (i.e, $k \geq e^e$) and the p-series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2\sqrt{k}} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^{\frac{1}{2}}}$ diverges, the series $\displaystyle\sum_{k=1}^{\infty}\left|\frac{(-1)^k\ln(\ln k)}{\sqrt{\ln k}+1}\right|$ diverges.

(e). Divergence. Since

$$\lim_{k\to\infty}\frac{k^k}{(k+1)^k} = \lim_{k\to\infty}\frac{1}{\left(\dfrac{k+1}{k}\right)^k} = \lim_{k\to\infty}\frac{1}{\left(1+\dfrac{1}{k}\right)^k} = \frac{1}{e},$$

the limit $\displaystyle\lim_{k\to\infty}\frac{(-1)^k k^k}{(k+1)^k}$ does not exist because it has two subsequential limits $\pm\dfrac{1}{e}$. By the divergence test, the series $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k k^k}{(k+1)^k}$ diverges. □

Question 2. [20 marks] Given that

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12},$$

determine how large $n \in \mathbb{N}$ can be chosen so that $\left|\frac{\pi^2}{12} - S_n\right| < 10^{-6}$, where S_n is the n th partial sum of the series.

Solution. Let $a_k = \frac{1}{k^2}$. Then $\{a_k\}$ is positive, monotone decreasing and $\lim_{k\to\infty} a_k = 0$. By the alternating series estimation, from

$$a_{n+1} = \frac{1}{(n+1)^2} < 10^{-6} \quad \Longrightarrow \quad (n+1)^2 > 10^6 \quad \Longrightarrow \quad n+1 > 1000$$

we have that when $n = 1000$, the error $\left|\frac{\pi^2}{12} - S_n\right|$ is less than 10^{-6}. □

Question 3. [20 marks] Determine the domain of the two-variable function defined by

$$f(x,y) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(\ln n)^y}{n^x}$$

converges.

Solution. **Case I.** $x > 0$. Let $a_n = \frac{(\ln n)^y}{n^x}$. Then $a_n \geq 0$. To see $\{a_n\}$ *eventually* monotone decreasing, let $g(t) = \frac{(\ln t)^y}{t^x}$. Then

$$g'(t) = \frac{y \cdot (\ln t)^{y-1} \cdot \frac{1}{t} \cdot t^x - (\ln t)^y \cdot x \cdot t^{x-1}}{t^{2x}} = \frac{x(\ln t)^{y-1}\left(\frac{y}{x} - \ln t\right)}{t^{x+1}}$$

is negative when $\ln t > \frac{y}{x}$ and $t > 1$, that is, $x > \max\{1, e^{\frac{y}{x}}\}$. Thus $\{a_n\}$ is eventually monotone decreasing. Now

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(\ln n)^y}{n^x} = 0$$

by the standard limits because $x > 0$. By the alternating series test, the series $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(\ln n)^y}{n^x}$ converges, that is, $f(x,y)$ is well-defined.

Case II. $x = 0$. In this case,

$$f(0,y) = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}(\ln n)^y}{1} = \sum_{n=2}^{\infty} (-1)^{n+1}(\ln n)^y.$$

When $y > 0$, $\lim_{n\to\infty} (\ln n)^y = \infty$, and when $y = 0$, $(\ln n)^0 = 1$. Thus, for $y \geq 0$, the limit $\lim_{n\to\infty} (-1)^{n+1}(\ln n)^y$ does not exists, and so, by the divergence test, the series $\sum_{n=2}^{\infty} (-1)^{n+1}(\ln n)^y$ divergence. For $y < 0$, let $a_n = (\ln n)^y$. Then $\{a_n\}$ is positive, monotone decreasing and $\lim_{n\to\infty} a_n = 0$. Thus the series $\sum_{n=2}^{\infty} (-1)^{n+1}(\ln n)^y$ converges by the alternating series test for $y < 0$.

Case III. $x < 0$. In this case, $-x > 0$. Since

$$\lim_{n\to\infty} \frac{1}{\left[\dfrac{(\ln n)^y}{n^x}\right]} = \lim_{n\to\infty} \frac{(\ln n)^{-y}}{n^{-x}} = 0$$

because $-x > 0$, $\displaystyle\lim_{n\to\infty} \frac{(\ln n)^y}{n^x} = \infty$ and so the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(\ln n)^y}{n^x}$ diverges by the divergence test.

In conclusion, the domain of the function $f(x,y) = \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(\ln n)^y}{n^x}$ is

$$\{(x,y) \mid x > 0, y \in \mathbb{R}\} \cup \{(0,y) \mid y < 0\}.$$

$\square$

Question 4. [20 marks] Suppose that the sequence $\{b_n\}$ is monotone decreasing with $\displaystyle\lim_{n\to\infty} b_n = 0$. If $\{a_n\}$ is a sequence satisfying $|a_n| \leq b_n - b_{n+1}$ for all n, prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely.

Proof. Let $S_n = \displaystyle\sum_{k=1}^{n} |a_k|$. Since $|a_k| \leq b_k - b_{k+1}$ for each k, we have

$$S_n = |a_1|+|a_2|+\cdots+|a_n| \leq (b_1-b_2)+(b_2-b_3)+(b_3-b_4)+\cdots+(b_n-b_{n+1}) = b_1-b_{n+1} \leq b_1$$

because each $b_n \geq 0$, which follows from that, since $\{b_n\}$ is monotone decreasing, $b_n \geq \inf\{b_n\} = \displaystyle\lim_{k\to\infty} b_k = 0$. Thus the sequence of partial sums of the positive series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ is bounded above by b_1, and so the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges. In other words, the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges absolutely. $\square$