Question 1 [20 marks]
For each of the following sequence of functions, determine whether converges point-wise to a function, and find the limiting function if it exists. Justify your answers.

(a) \(\left\{\left(1 - \frac{x^2}{n}\right)^{nx}\right\}, \ x \in \mathbb{R}\).

(b) \(\{(\cos x)^{2n}\}, \ x \in \mathbb{R}\).

(c) \(\left\{\frac{\sin nx}{\cos nx + nx}\right\}, \ x \in [1, +\infty)\).

(d) \(\{f_n(x)\}, \ f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1 + x^2)^k}, \ x \in \mathbb{R}\).

Solution. (a).

\[
\lim_{n \to \infty} \left(1 - \frac{x^2}{n}\right)^{nx} = \lim_{n \to \infty} \left[\left(1 - \frac{x^2}{n}\right)^n\right]^x = \left(e^{-x^2}\right)^x = e^{-x^3}.
\]

(b). When \(x = k\pi\), then \(\cos x = (-1)^k\) and so \(\lim_{n \to \infty} (\cos x)^{2n} = \lim_{n \to \infty} 1 = 1\) for \(x = k\pi\). When \(x \neq k\pi\), the \(|\cos x| < 1\) and so \(\lim_{n \to \infty} (\cos x)^{2n} = 0\) in this case. Thus

\[
\lim_{n \to \infty} (\cos x)^{2n} = \begin{cases} 
0 & x \neq 0, \pm \pi, \pm 2\pi, \pm 3\pi, \cdots \\
1 & x = k\pi \text{ for some } k \in \mathbb{Z}.
\end{cases}
\]

(c).

\[
\lim_{n \to \infty} \frac{\sin nx}{\cos nx + nx} = \lim_{n \to \infty} \frac{\sin nx}{\cos nx} = 0,
\]

where \(\lim_{n \to \infty} \frac{\sin nx}{n} = \lim_{n \to \infty} \frac{\cos nx}{n} = 0\), by the Squeeze Theorem, because

\[
0 \leq \left|\frac{\sin nx}{n}\right|, \left|\frac{\cos nx}{n}\right| \leq \frac{1}{n}
\]

and \(\lim_{n \to \infty} \frac{1}{n} = 0\).

(d). When \(x = 0\), \(f_n(x) = 0\). In this case \(\lim_{n \to \infty} f_n(x) = 0\). When \(x \neq 0\),

\[
f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1 + x^2)^k}
\]

\[
= x^2 \left[1 + \frac{1}{1 + x^2} + \cdots + \left(\frac{1}{1 + x^2}\right)^n\right]
\]
\[
\begin{align*}
&= x^2 \frac{1 - \left(\frac{1}{1 + x^2}\right)^{n+1}}{1 - \frac{1}{1 + x^2}} \\
\lim_{n \to \infty} f_n(x) &= \lim_{n \to \infty} x^2 \frac{1 - \left(\frac{1}{1 + x^2}\right)^{n+1}}{1 - \frac{1}{1 + x^2}} = x^2 \frac{1}{1 - \frac{1}{1 + x^2}} = 1 + x^2
\end{align*}
\]

because the positive number \(\frac{1}{1 + x^2}\) < 1 for \(x \neq 0\). Thus
\[
\lim_{n \to \infty} f_n(x) = \begin{cases} 
0 & x = 0 \\
1 + x^2 & x \neq 0.
\end{cases}
\]

\[\square\]

**Question 2.** [30 marks] Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

(a) \(F_n(x) = \frac{x^n}{1 + x^n}, \ x \in [0, \frac{1}{2}]\).

(b) \(F_n(x) = \frac{x^n}{1 + x^n}, \ x \in [0, 1]\).

(c) \(F_n(x) = x + \frac{\sin nx}{\frac{n}{x}}, \ x \in [-a, a], \ a > 0\).

(d) \(F_n(x) = x + \frac{\sin nx}{x}, \ x \in \mathbb{R}\).

(e) \(F_n(x) = \frac{x^n \sin nx}{1 + x^n}, \ x \in [0, \frac{1}{2}]\).

(f) \(F_n(x) = nx \left(1 - x^2\right)^n, \ x \in [0, 1]\).

**Solution.**

(a). Uniform convergence.

\[
F(x) = \lim_{n \to \infty} F_n(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n} = \frac{0}{1+0} = 0 \quad 0 \leq x \leq \frac{1}{2}
\]

\[
T_n = \sup_{0 \leq x \leq \frac{1}{2}} |F_n(x) - F(x)|
\]

\[
= \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n}{1 + x^n}
\]

\[
\leq \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n}{1} = \left(\frac{1}{2}\right)^n
\]

Since \(\lim_{n \to \infty} \left(\frac{1}{2}\right)^n = 0\), \(\lim_{n \to \infty} T_n = 0\) by the Squeeze theorem and so the sequence of the functions converges uniformly.

(b). NOT uniform convergence.

\[
F(x) = \lim_{n \to \infty} F_n(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n} = \begin{cases} 
0 & 0 \leq x < 1 \\
\frac{1}{1+1} = \frac{1}{2} & x = 1.
\end{cases}
\]
Since each $F_n(x) = \frac{x^n}{1+x^n}$ is continuous on $[0, 1]$ and its limiting function $F(x)$ is NOT continuous on $[0, 1]$, the sequence of functions does not converge uniformly on $[0, 1]$.

For (c) and (d), $F(x) = \lim_{n \to \infty} x + \frac{x}{n} \sin nx = x$ because $\lim_{n \to \infty} \frac{\sin nx}{n} = 0$.

(c). Uniform convergence.

$$T_n = \sup_{x \in [-a,a]} |F_n(x) - F(x)| = \sup_{x \in [-a,a]} \left|x + \frac{x}{n} \sin nx - x\right| = \sup_{x \in [-a,a]} \left|x \sin nx\right| \leq \sup_{x \in [-a,a]} \left|x\right| \cdot \frac{a}{n} = \frac{a}{n}.$$  

Since $\lim_{n \to \infty} \frac{a}{n} = 0$, by the Squeeze Theorem, $\lim_{n \to \infty} T_n = 0$ and so the sequence of functions converges uniformly.

(d). NOT uniform convergence.

$$T_n = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| = \sup_{x \in \mathbb{R}} \left|x + \frac{x}{n} \sin nx - x\right| = \sup_{x \in \mathbb{R}} \frac{|x| \cdot |\sin nx|}{n}.$$  

By choosing $x = 2n\pi + \frac{\pi}{2n}$, we have

$$T_n \geq \frac{|2n\pi + \frac{\pi}{2n}| \cdot \left|\sin (2n\pi + \frac{\pi}{2})\right|}{n} = \frac{(2n\pi + \frac{\pi}{2n}) \cdot \left|\sin (2n^2\pi + \frac{\pi}{2})\right|}{n} = \frac{(2n\pi + \frac{\pi}{2n}) \cdot 1}{n} = \frac{2n\pi + \frac{\pi}{2n}}{n} \geq 2\pi.$$  

Thus $T_n$ does not tend to zero as $n$ tends to $\infty$, and so the sequence of functions does not converge uniformly.

(e). Uniform convergence. Since

$$\left|\frac{x^n \sin nx}{1+x^n}\right| \leq \left|\frac{x^n}{1+x^n}\right|$$

and $\lim_{n \to \infty} \frac{x^n}{1+x^n} = 0$ for $0 \leq x \leq \frac{1}{2}$, we have

$$F(x) = \lim_{n \to \infty} F_n(x) = \lim_{n \to \infty} \frac{x^n \sin nx}{1+x^n} = 0.$$
by the Squeeze Theorem. Note that

\[ T_n = \sup_{0 \leq x \leq \frac{1}{2}} |F_n(x) - F(x)| = \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n |\sin nx|}{1 + x^n} \]

\[ \leq \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n}{1 + x^n} \]

\[ \leq \sup_{0 \leq x \leq \frac{1}{2}} \frac{x^n}{1} \]

\[ = \left( \frac{1}{2} \right)^n. \]

Since \( \lim_{n \to \infty} \left( \frac{1}{2} \right)^n = 0 \), \( \lim_{n \to \infty} T_n = 0 \) by the Squeeze theorem and so the sequence of the functions converges uniformly.

(f). NOT uniform convergence. When \( x = 0, 1 \), \( F_n(x) = 0 \) and so \( F(x) = \lim_{n \to \infty} F_n(x) = 0 \) for \( x = 0, 1 \). When \( 0 < x < 1 \), then \( 0 < 1 - x^2 < 1 \) or \( \frac{1}{1 - x^2} > 1 \). Thus, for \( 0 < x < 1 \),

\[ F(x) = \lim_{n \to \infty} F_n(x) = \lim_{n \to \infty} n x (1 - x^2)^n = \lim_{n \to \infty} \frac{n}{(1 - x^2)^n} \cdot x = 0 \]

by the Standard limits. Thus \( F(x) = 0 \) for all \( 0 \leq x \leq 1 \). Now

\[ T_n = \sup_{0 \leq x \leq 1} |F_n(x) - F(x)| = \sup_{0 \leq x \leq 1} F_n(x) = \sup_{0 \leq x \leq 1} nx (1 - x^2)^n. \]

We find the maximum of \( F_n(x) \) on \([0, 1]\).

\[ F'(x) = n (1 - x^2)^n + nx \cdot n (1 - x^2)^{n-1} \cdot (-2x) = 0 \]

\[ \implies n (1 - x^2)^{n-1} (1 - x^2 - 2nx^2) = n (1 - x^2)^{n-1} (1 - (2n+1)x^2) = 0 \]

\[ \implies x = 0, 1, \frac{1}{\sqrt{2n+1}}. \]

Now

\[ F_n(0) = F_n(1) = 0, \quad F_n \left( \frac{1}{\sqrt{2n+1}} \right) = n \cdot \frac{1}{\sqrt{2n+1}} \left( 1 - \frac{1}{2n+1} \right)^n \]

Thus \( T_n = F_n \left( \frac{1}{\sqrt{2n+1}} \right) = n \cdot \frac{1}{\sqrt{2n+1}} \left( 1 - \frac{1}{2n+1} \right)^n \) and

\[ \lim_{n \to \infty} T_n = \lim_{n \to \infty} n \cdot \frac{1}{\sqrt{2n+1}} \left( 1 - \frac{1}{2n+1} \right)^n \]

\[ \lim_{n \to \infty} \sqrt{n} \cdot \frac{1}{\sqrt{2 + 1/n}} \cdot \left[ \left( 1 + \frac{-1}{2n+1} \right)^{2n+1} \right]^{\frac{n}{2n+1}} \]

\[ = \infty \cdot \frac{1}{2} \cdot (e^{-1})^{\frac{1}{2}} = \infty. \]

Hence the sequence of functions does not converge uniformly. \( \square \)
Question 3. [30 marks] Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(a) $\sum_{k=1}^{\infty} \frac{k \sin kx}{k^3 + x^2}$, $x \in [0, \infty)$.

(b) $\sum_{k=1}^{\infty} e^{-kx}x^k$, $x \in [0, \infty)$.

(c) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k + x}$, $x \in [0, \infty)$.

(d) $\sum_{k=1}^{\infty} \frac{x^k}{1 + k (\ln k)^2}$, $x \in [-1, 1]$.

(e) $\sum_{n=0}^{\infty} \left( \frac{1}{nx + 2} - \frac{1}{(n+1)x + 2} \right)$, $x \in [0, 1]$.

(f) $\sum_{k=1}^{\infty} \left( \frac{x}{2} \right)^k$, $x \in (-2, 2)$.

Solution. (a). Uniform convergence. Since

$$\left| \frac{k \sin kx}{k^3 + x^2} \right| \leq \frac{k}{k^3 + x^2} \leq \frac{k}{k^3} = \frac{1}{k^2}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the series of functions converges uniformly by the Weierstrass $M$-test.

(b). Uniform convergence. Let $f_k(x) = e^{-kx}x^k$.

$$f'(x) = (-k) \cdot e^{-kx} \cdot x^k + e^{-kx} \cdot k \cdot x^{k-1} = ke^{-kx}x^{k-1}(1-x) = 0$$

$\implies x = 0, 1$.

Since $f_k(0) = 0$ and $\lim_{x \to \infty} \frac{x^k}{e^{kx}} = 0$, the maximum of $|f_k(x)| = f_k(x)$ on $[0, \infty)$ is $f_k(1) = e^{-k}$, that is,

$$\left| e^{-kx}x^k \right| \leq e^{-k} \quad x \in [0, \infty).$$

Since $\sum_{k=1}^{\infty} e^{-k} = \sum_{k=1}^{\infty} \left( \frac{1}{e} \right)^k$ converges by the geometric series, the series of functions converges uniformly by the Weierstrass $M$-test.

(c). Uniform convergence. Let $a_k = \frac{1}{k + x}$. Then, for $x \geq 0$, $a_k$ is positive, monotone decreasing and $\lim_{k \to \infty} a_k = 0$. Let $S(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k + x}$ and let $S_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k + x}$. By the alternating series estimation,

$$|S_n(x) - S(x)| \leq a_{n+1} = \frac{1}{n + 1 + x}.$$
Thus
\[ T_n = \sup_{x \geq 0} |S_n(x) - S(x)| \leq \sup_{x \geq 0} \frac{1}{n + 1 + x} \leq \frac{1}{n + 1}. \]

Since \( \lim_{n \to \infty} \frac{1}{n + 1} = 0 \), by the Squeeze Theorem, \( \lim_{n \to \infty} T_n = 0 \) and so the series of functions converges uniformly.

(d). Uniform convergence. Note that
\[
\left| \frac{x^k}{1 + k(\ln k)^2} \right| \leq \frac{1}{1 + k(\ln k)^2} \quad x \in [0, 1].
\]

Since \( \frac{1}{1 + k(\ln k)^2} \leq \frac{1}{k(\ln k)^2} \) for \( k \geq 2 \) and \( \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \) converges by the integral test, the series \( \sum_{k=1}^{\infty} \frac{1}{1 + k(\ln k)^2} \) converges by the comparison test. From the inequality above, the series of functions converges uniformly by the Weierstrass \( M \)-test.

(e). NOT uniform convergence. Consider the partial sums
\[
S_n(x) = \sum_{k=0}^{n} \left( \frac{1}{kx + 2} - \frac{1}{(k+1)x + 2} \right) = \left( \frac{1}{2} - \frac{1}{x + 2} \right) + \left( \frac{1}{x + 2} - \frac{1}{2x + 2} \right) + \cdots + \left( \frac{1}{nx + 2} - \frac{1}{(n+1)x + 2} \right) = \frac{1}{2} - \frac{1}{(n+1)x + 2}.
\]

Thus
\[
S(x) = \sum_{n=0}^{\infty} \left( \frac{1}{nx + 2} - \frac{1}{(n+1)x + 2} \right) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \left( \frac{1}{2} - \frac{1}{(n+1)x + 2} \right) = \begin{cases} 
0 & x = 0 \\
\frac{1}{2} & x \neq 0.
\end{cases}
\]

Since each \( S_n(x) \) is continuous on \([0, 1]\) and \( S(x) \) is NOT continuous on \([0, 1]\), the series of functions does not converge uniformly.

(f). NOT uniform convergence. Note that
\[
T_n = \sup_{-2 < x < 2} \left| \sum_{k=n+1}^{\infty} \left( \frac{x}{2} \right)^k \right| = \sup_{-2 < x < 2} \left| \left( \frac{x}{2} \right)^{n+1} \left[ 1 + \frac{x}{2} + \left( \frac{x}{2} \right)^2 + \cdots \right] \right|
\]
\[
= \sup_{-2 < x < 2} \frac{\left( \frac{|x|}{2} \right)^{n+1}}{1 - \frac{x}{2}} \quad \text{let } x \to 2^{-} \quad + \infty.
\]
Since $T_n$ does not tend to 0 as $n$ tends to $\infty$, the series of functions does not converge uniformly. \qed

**Question 4.** [20 marks] Show that each of the following series converges uniformly on $[a, \infty)$ for any $a > 0$, but does not converge uniformly on $(0, \infty)$.

(a) $\sum_{n=0}^{\infty} \frac{1}{1 + n^2 x}$.

(b) $\sum_{k=1}^{\infty} \frac{1}{k^{1+x}}$.

**Proof.** (a). On $[a, \infty)$ with $a > 0$, since

$$\left| \frac{1}{1 + n^2 x} \right| \leq \frac{1}{1 + n^2 a} \leq \frac{1}{an^2}$$

and $\sum_{n=1}^{\infty} \frac{1}{an^2} = \frac{1}{a} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series of functions converges uniformly on $[a, \infty)$ by the Weierstrass $M$-test.

On $(0, \infty)$,

$$T_n = \sup_{x > 0} \left| \sum_{k=n+1}^{\infty} \frac{1}{1 + k^2 x} \right|$$

$$= \sup_{x > 0} \sum_{k=n+1}^{\infty} \frac{1}{1 + k^2 x}$$

$$\geq \sup_{x > 0} \frac{1}{1 + (n + 1)^2 x} = 1$$

because the function $\frac{1}{1 + (n + 1)^2 x}$ is monotone decreasing on $(0, 1)$ with

$$\lim_{x \to 0} \frac{1}{1 + (n + 1)^2 x} = 1$$

Thus $T_n$ does not tend to 0 as $n$ tends to $\infty$, and so the series of functions $\sum_{n=0}^{\infty} \frac{1}{1 + n^2 x}$ does not converge uniformly on $(0, \infty)$. (Another solution: You can prove by contradiction using the Cauchy Criterion.)

(b). On $[a, \infty)$ with $a > 0$, since

$$\left| \frac{1}{n^{1+x}} \right| \leq \frac{1}{n^{1+a}}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{1+a}}$ converges, the series of functions converges uniformly on $[a, \infty)$ by the Weierstrass $M$-test.

On $(0, \infty)$,

$$T_n = \sup_{x > 0} \left| \sum_{k=n+1}^{\infty} \frac{1}{k^{1+x}} \right|$$
\[ \sum_{k=n+1}^{\infty} \frac{1}{k^{1+x}} \geq \sup_{x>0} \left( \frac{1}{(n+1)^{1+x}} + \frac{1}{(n+2)^{1+x}} + \cdots + \frac{1}{(2n)^{1+x}} \right) \]

\[ \geq \sup_{x>0} \left( \frac{1}{(2n)^{1+x}} + \frac{1}{(2n)^{1+x}} + \cdots + \frac{1}{(2n)^{1+x}} \right) \]

\[ = \sup_{x>0} \frac{n}{(2n)^{1+x}} = \sup_{x>0} \frac{1}{2 \cdot (2n)^x} \]

Since the function \( \frac{1}{2 \cdot (2n)^x} \) is monotone decreasing on \((0, \infty)\) with \( \lim_{x \to 0} \frac{1}{2 \cdot (2n)^x} = \frac{1}{2} \),
we have \( \sup_{x>0} \frac{1}{2 \cdot (2n)^x} = \frac{1}{2} \) and so

\[ T_n \geq \frac{1}{2} \]

It follows that \( T_n \) does not tend to 0 as \( n \) tends to \( \infty \), and so the series of functions
\[ \sum_{k=1}^{\infty} \frac{1}{k^{1+x}} \]
does not converge uniformly on \((0, \infty)\). (Another solution: You can prove by contradiction using the Cauchy Criterion.)