Take-home Exam 5

Question 1 [20 marks]
For each of the following sequence of functions, determine whether is converges pointwise to a function, and find the limiting function if it exists. Justify your answers.

(a) $\left\{\left(1-\frac{x^2}{n}\right)^{nx}\right\}, \quad x\in\mathbb{R}.$

(b) $\{(\cos x)^{2n}\}, \quad x\in\mathbb{R}.$

(c) $\left\{\frac{\sin nx}{\cos nx+nx}\right\}, \quad x\in[1,+\infty).$

(d) $\{f_n(x)\}, \quad f_n(x)=\sum_{k=0}^{n}\frac{x^2}{(1+x^2)^k}, \quad x\in\mathbb{R}.$

Solution. (a).

$$\lim_{n\to\infty}\left(1-\frac{x^2}{n}\right)^{nx}=\lim_{n\to\infty}\left[\left(1-\frac{x^2}{n}\right)^{n}\right]^{x}=\left(e^{-x^2}\right)^{x}=e^{-x^3}.$$

(b). When $x=k\pi$, then $\cos x=(-1)^k$ and so $\lim_{n\to\infty}(\cos x)^{2n}=\lim_{n\to\infty}1=1$ for $x=k\pi$. When $x\neq k\pi$, the $|\cos x|<1$ and so $\lim_{n\to\infty}(\cos x)^{2n}=0$ in this case. Thus

$$\lim_{n\to\infty}(\cos x)^{2n}=\begin{cases}0 & x\neq 0,\pm\pi,\pm2\pi,\pm3\pi,\cdots\\ 1 & x=k\pi \quad \text{for some} \quad k\in\mathbb{Z}.\end{cases}$$

(c).

$$\lim_{n\to\infty}\frac{\sin nx}{\cos nx+nx}=\lim_{n\to\infty}\frac{\frac{\sin nx}{n}}{\frac{\cos nx}{n}+x}=0,$$

where $\lim_{n\to\infty}\frac{\sin nx}{n}=\lim_{n\to\infty}\frac{\cos nx}{n}=0$, by the Squeeze Theorem, because

$$0\le\left|\frac{\sin nx}{n}\right|,\left|\frac{\cos nx}{n}\right|\le\frac{1}{n}$$

and $\lim_{n\to\infty}\frac{1}{n}=0.$

(d). When $x=0$, $f_n(x)=0$. In this case $\lim_{n\to\infty}f_n(x)=0$. When $x\neq0$,

$$f_n(x)=\sum_{k=0}^{n}\frac{x^2}{(1+x^2)^k}$$

$$=x^2\left[1+\frac{1}{1+x^2}+\cdots+\left(\frac{1}{1+x^2}\right)^n\right]$$

$$= x^2 \frac{1 - \left(\dfrac{1}{1+x^2}\right)^{n+1}}{1 - \dfrac{1}{1+x^2}}$$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^2 \frac{1 - \left(\dfrac{1}{1+x^2}\right)^{n+1}}{1 - \dfrac{1}{1+x^2}} = x^2 \frac{1}{1 - \dfrac{1}{1+x^2}} = 1 + x^2$$

because the positive number $\dfrac{1}{1+x^2} < 1$ for $x \neq 0$. Thus

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x = 0 \\ 1 + x^2 & x \neq 0. \end{cases}$$

□

Question 2. [30 marks] Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

(a) $F_n(x) = \dfrac{x^n}{1+x^n}, \quad x \in [0, \frac{1}{2}]$.

(b) $F_n(x) = \dfrac{x^n}{1+x^n}, \quad x \in [0, 1]$.

(c) $F_n(x) = x + \dfrac{x}{n} \sin nx, \quad x \in [-a, a], \quad a > 0$.

(d) $F_n(x) = x + \dfrac{x}{n} \sin nx, \quad x \in \mathbb{R}$.

(e) $F_n(x) = \dfrac{x^n \sin nx}{1+x^n}, \quad x \in [0, \frac{1}{2}]$.

(f) $F_n(x) = nx\left(1 - x^2\right)^n, \quad x \in [0, 1]$.

Solution. (a). Uniform convergence.

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0 \qquad 0 \le x \le \frac{1}{2}$$

$$\begin{aligned} T_n &= \sup_{0\le x\le \frac{1}{2}} |F_n(x) - F(x)| \\ &= \sup_{0\le x\le \frac{1}{2}} \frac{x^n}{1+x^n} \\ &\le \sup_{0\le x\le \frac{1}{2}} \frac{x^n}{1} = \left(\frac{1}{2}\right)^n \end{aligned}$$

Since $\lim_{n\to\infty} \left(\dfrac{1}{2}\right)^n = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze theorem and so the sequence of the functions converges uniformly.

(b). NOT uniform convergence.

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \begin{cases} 0 & 0 \le x < 1 \\ \frac{1}{1+1} = \frac{1}{2} & x = 1. \end{cases}$$

Since each $F_n(x) = \dfrac{x^n}{1+x^n}$ is continuous on $[0,1]$ and its limiting function $F(x)$ is NOT continuous on $[0,1]$, the sequence of functions does not converge uniformly on $[0,1]$.

For (c) and (d), $F(x) = \lim_{n\to\infty} x + \dfrac{x}{n}\sin nx = x$ because $\lim_{n\to\infty} \dfrac{\sin nx}{n} = 0$.

(c). Uniform convergence.

$$\begin{aligned} T_n &= \sup_{x\in[-a,a]} |F_n(x) - F(x)| \\ &= \sup_{x\in[-a,a]} \left| x + \frac{x}{n}\sin nx - x \right| \\ &= \sup_{x\in[-a,a]} \left| \frac{x}{n}\sin nx \right| \\ &\le \sup_{x\in[-a,a]} \frac{|x|}{n} = \frac{a}{n}. \end{aligned}$$

Since $\lim_{n\to\infty} \dfrac{a}{n} = 0$, by the Squeeze Theorem, $\lim_{n\to\infty} T_n = 0$ and so the sequence of functions converges uniformly.

(d). NOT uniform convergence.

$$T_n = \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = \sup_{x\in\mathbb{R}} \left| x + \frac{x}{n}\sin nx - x \right| = \sup_{x\in\mathbb{R}} \frac{|x| \cdot |\sin nx|}{n}$$

By choosing $x = 2n\pi + \dfrac{\pi}{2n}$, we have

$$\begin{aligned} T_n &\ge \frac{\left|2n\pi + \frac{\pi}{2n}\right| \cdot \left|\sin n(2n\pi + \frac{\pi}{2n})\right|}{n} \\ &= \frac{(2n\pi + \frac{\pi}{2n}) \cdot \left|\sin(2n^2\pi + \frac{\pi}{2})\right|}{n} \\ &= \frac{(2n\pi + \frac{\pi}{2n}) \cdot 1}{n} \\ &= 2\pi + \frac{\pi}{2n^2} \ge 2\pi. \end{aligned}$$

Thus T_n does not tend to zero as n tends to ∞, and so the sequence of functions does not converge uniformly.

(e). Uniform convergence. Since

$$\left| \frac{x^n \sin nx}{1+x^n} \right| \le \left| \frac{x^n}{1+x^n} \right|$$

and $\lim_{n\to\infty} \dfrac{x^n}{1+x^n} = 0$ for $0 \le x \le \dfrac{1}{2}$, we have

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{x^n \sin nx}{1+x^n} = 0$$

by the Squeeze Theorem. Note that

$$
\begin{aligned}
T_n = \sup_{0\le x\le \frac{1}{2}} |F_n(x) - F(x)| = \sup_{0\le x\le \frac{1}{2}} \frac{x^n|\sin nx|}{1+x^n} \\
\le \sup_{0\le x\le \frac{1}{2}} \frac{x^n}{1+x^n} \\
\le \sup_{0\le x\le \frac{1}{2}} \frac{x^n}{1} \\
= \left(\frac{1}{2}\right)^n .
\end{aligned}
$$

Since $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze theorem and so the sequence of the functions converges uniformly.

(f). NOT uniform convergence. When $x = 0, 1$, $F_n(x) = 0$ and so $F(x) = \lim_{n\to\infty} F_n(x) = 0$ for $x = 0, 1$. When $0 < x < 1$, then $0 < 1 - x^2 < 1$ or $\frac{1}{1-x^2} > 1$. Thus, for $0 < x < 1$,

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} nx\left(1-x^2\right)^n = \lim_{n\to\infty} \frac{n}{\left(\frac{1}{1-x^2}\right)^n}\cdot x = 0$$

by the Standard limits. Thus $F(x) = 0$ for all $0 \le x \le 1$. Now

$$T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} F_n(x) = \sup_{0\le x\le 1} nx\left(1-x^2\right)^n .$$

We find the maximum of $F_n(x)$ on $[0, 1]$.

$$
\begin{aligned}
F_n'(x) = n\left(1-x^2\right)^n + nx\cdot n\left(1-x^2\right)^{n-1}\cdot(-2x) = 0 \\
\Longrightarrow n\left(1-x^2\right)^{n-1}\left(1-x^2-2nx^2\right) = n\left(1-x^2\right)^{n-1}\left(1-(2n+1)x^2\right) = 0 \\
\Longrightarrow x = 0, 1, \frac{1}{\sqrt{2n+1}}.
\end{aligned}
$$

Now

$$F_n(0) = F_n(1) = 0, \qquad F_n\left(\frac{1}{\sqrt{2n+1}}\right) = n\cdot\frac{1}{\sqrt{2n+1}}\left(1-\frac{1}{2n+1}\right)^n$$

Thus $T_n = F_n\left(\frac{1}{\sqrt{2n+1}}\right) = n\cdot\frac{1}{\sqrt{2n+1}}\left(1-\frac{1}{2n+1}\right)^n$ and

$$
\begin{aligned}
\lim_{n\to\infty} T_n = \lim_{n\to\infty} n\cdot\frac{1}{\sqrt{2n+1}}\left(1-\frac{1}{2n+1}\right)^n \\
= \lim_{n\to\infty} \sqrt{n}\cdot\frac{1}{\sqrt{2+1/n}}\cdot\left[\left(1+\frac{-1}{2n+1}\right)^{2n+1}\right]^{\frac{n}{2n+1}} \\
= \infty\cdot\frac{1}{2}\cdot\left(e^{-1}\right)^{\frac{1}{2}} = \infty.
\end{aligned}
$$

Hence the sequence of functions does not converge uniformly. □

Question 3. [30 marks] Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k \sin kx}{k^3 + x^2}, \quad x \in [0, \infty).$

(b) $\displaystyle\sum_{k=1}^{\infty} e^{-kx} x^k, \quad x \in [0, \infty).$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+x}, \quad x \in [0, \infty).$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{1 + k(\ln k)^2}, \quad x \in [-1, 1].$

(e) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{nx+2} - \frac{1}{(n+1)x+2} \right), \quad x \in [0, 1].$

(f) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{x}{2} \right)^k, \quad x \in (-2, 2).$

Solution. (a). Uniform convergence. Since

$$\left| \frac{k \sin kx}{k^3 + x^2} \right| \leq \frac{k}{k^3 + x^2} \leq \frac{k}{k^3} = \frac{1}{k^2}$$

and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, the series of functions converges uniformly by the Weierstrass M-test.

(b). Uniform convergence. Let $f_k(x) = e^{-kx} x^k$.

$$f'(x) = (-k) \cdot e^{-kx} \cdot x^k + e^{-kx} \cdot k \cdot x^{k-1} = k e^{-kx} x^{k-1}(1 - x) = 0$$
$$\Longrightarrow x = 0, 1.$$

Since $f_k(0) = 0$ and $\displaystyle\lim_{x \to \infty} f_k(x) = \lim_{x \to \infty} \frac{x^k}{e^{kx}} = 0$, the maximum of $|f_k(x)| = f_k(x)$ on $[0, \infty)$ is $f_k(1) = e^{-k}$, that is,

$$\left| e^{-kx} x^k \right| \leq e^{-k} \qquad x \in [0, \infty).$$

Since $\displaystyle\sum_{k=1}^{\infty} e^{-k} = \sum_{k=1}^{\infty} \left(\frac{1}{e} \right)^k$ converges by the geometric series, the series of functions converges uniformly by the Weierstrass M-test.

(c). Uniform convergence. Let $a_k = \dfrac{1}{k+x}$. Then, for $x \geq 0$, a_k is positive, monotone decreasing and $\displaystyle\lim_{k \to \infty} a_k = 0$. Let $S(x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+x}$ and let $S_n(x) = \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+x}$. By the alternating series estimation,

$$|S_n(x) - S(x)| \leq a_{n+1} = \frac{1}{n+1+x}.$$

Thus

$$T_n = \sup_{x\geq 0} |S_n(x) - S(x)| \leq \sup_{x\geq 0} \frac{1}{n+1+x} \leq \frac{1}{n+1}.$$

Since $\lim_{n\to\infty} \frac{1}{n+1} = 0$, by the Squeeze Theorem, $\lim_{n\to\infty} T_n = 0$ and so the series of functions converges uniformly.

(d). Uniform convergence. Note that

$$\left|\frac{x^k}{1+k(\ln k)^2}\right| \leq \frac{1}{1+k(\ln k)^2} \qquad x \in [0,1].$$

Since $\frac{1}{1+k(\ln k)^2} \leq \frac{1}{k(\ln k)^2}$ for $k \geq 2$ and $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges by the integral test, the series $\sum_{k=1}^{\infty} \frac{1}{1+k(\ln k)^2}$ converges by the comparison test. From the inequality above, the series of functions converges uniformly by the Weierstrass M-test.

(e). NOT uniform convergence. Consider the partial sums

$$S_n(x) = \sum_{k=0}^{n} \left(\frac{1}{kx+2} - \frac{1}{(k+1)x+2}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{x+2}\right) + \left(\frac{1}{x+2} - \frac{1}{2x+2}\right) + \cdots + \left(\frac{1}{nx+2} - \frac{1}{(n+1)x+2}\right)$$

$$= \frac{1}{2} - \frac{1}{(n+1)x+2}.$$

Thus

$$S(x) = \sum_{n=0}^{\infty} \left(\frac{1}{nx+2} - \frac{1}{(n+1)x+2}\right) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{(n+1)x+2}\right)$$

$$= \begin{cases} 0 & x = 0 \\ \frac{1}{2} & x \neq 0. \end{cases}$$

Since each $S_n(x)$ is continuous on $[0,1]$ and $S(x)$ is NOT continuous on $[0,1]$, the series of functions does not converge uniformly.

(f). NOT uniform convergence. Note that

$$T_n = \sup_{-2<x<2} \left|\sum_{k=n+1}^{\infty} \left(\frac{x}{2}\right)^k\right|$$

$$= \sup_{-2<x<2} \left|\left(\frac{x}{2}\right)^{n+1} \left[1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \cdots\right]\right|$$

$$= \sup_{-2<x<2} \frac{\left(\frac{|x|}{2}\right)^{n+1}}{1-\frac{x}{2}}$$

$$\overset{\text{let } x\to 2^-}{=\!=\!=} +\infty.$$

Since T_n does not tend to 0 as n tends to ∞, the series of functions does not converge uniformly. $\square$

Question 4. [20 marks] Show that each of the following series converges uniformly on $[a,\infty)$ for any $a>0$, but does not converge uniformly on $(0,\infty)$.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{1+n^2x}$.

(b) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{1+x}}$.

Proof. (a). On $[a,\infty)$ with $a>0$, since

$$\left|\frac{1}{1+n^2x}\right|\le\frac{1}{1+n^2a}\le\frac{1}{an^2}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{an^2}=\frac{1}{a}\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, the series of functions converges uniformly on $[a,\infty)$ by the Weierstrass M-test.

On $(0,\infty)$,

$$\begin{aligned}T_n&=\sup_{x>0}\left|\sum_{k=n+1}^{\infty}\frac{1}{1+k^2x}\right|\\&=\sup_{x>0}\sum_{k=n+1}^{\infty}\frac{1}{1+k^2x}\\&\ge\sup_{x>0}\frac{1}{1+(n+1)^2x}=1\end{aligned}$$

because the function $\dfrac{1}{1+(n+1)^2x}$ is monotone decreasing on $(0,1)$ with

$$\lim_{x\to0}\frac{1}{1+(n+1)^2x}=1$$

Thus T_n does not tend to 0 as n tends to ∞, and so the series of functions $\displaystyle\sum_{n=0}^{\infty}\frac{1}{1+n^2x}$ does not converge uniformly on $(0,\infty)$. (Another solution: You can prove by contradiction using the Cauchy Criterion.)

(b). On $[a,\infty)$ with $a>0$, since

$$\left|\frac{1}{n^{1+x}}\right|\le\frac{1}{n^{1+a}}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1+a}}$ converges, the series of functions converges uniformly on $[a,\infty)$ by the Weierstrass M-test.

On $(0,\infty)$,

$$T_n=\sup_{x>0}\left|\sum_{k=n+1}^{\infty}\frac{1}{k^{1+x}}\right|$$

$$
\begin{aligned}
&= \sup_{x>0} \sum_{k=n+1}^{\infty} \frac{1}{k^{1+x}} \\
&\geq \sup_{x>0} \left(\frac{1}{(n+1)^{1+x}} + \frac{1}{(n+2)^{1+x}} + \cdots + \frac{1}{(2n)^{1+x}} \right) \\
&\geq \sup_{x>0} \left(\frac{1}{(2n)^{1+x}} + \frac{1}{(2n)^{1+x}} + \cdots + \frac{1}{(2n)^{1+x}} \right) \\
&= \sup_{x>0} \frac{n}{(2n)^{1+x}} = \sup_{x>0} \frac{1}{2 \cdot (2n)^x}
\end{aligned}
$$

Since the function $\frac{1}{2 \cdot (2n)^x}$ is monotone decreasing on $(0, \infty)$ with $\lim_{x \to 0} \frac{1}{2 \cdot (2n)^x} = \frac{1}{2}$, we have $\sup_{x>0} \frac{1}{2 \cdot (2n)^x} = \frac{1}{2}$ and so

$$T_n \geq \frac{1}{2}.$$

It follows that T_n does not tend to 0 as n tends to ∞, and so the series of functions $\sum_{k=1}^{\infty} \frac{1}{k^{1+x}}$ does not converge uniformly on $(0, \infty)$. (Another solution: You can prove by contradiction using the Cauchy Criterion.) $\square$