

Take-home Exam 6

Question 1 [20 marks]

Find the radius of convergence of each of the following power series:

- (a) $\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k$.
- (b) $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right)^{k^2} (x-1)^k$.
- (c) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!} x^k$.
- (d) $\sum_{n=0}^{\infty} n \left(\frac{x}{2}\right)^{n^2}$.

Solution. (a). Note that $\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k = \sum_{k=1}^{\infty} \frac{3^k \cdot 2^k}{k^3} \left(x + \frac{1}{2}\right)^k$. Let $a_k = \frac{3^k \cdot 2^k}{k^3}$.

Then the radius of convergence

$$R = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k \rightarrow \infty} \left(\frac{3^k \cdot 2^k}{k^3}\right)^{\frac{1}{k}}} = \frac{1}{\lim_{k \rightarrow \infty} \frac{6}{(\sqrt[k]{k})^3}} = \frac{1}{6}.$$

(b). The radius of convergence

$$R = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k \rightarrow \infty} \left[\left(1 - \frac{1}{k}\right)^{k^2}\right]^{\frac{1}{k}}} = \frac{1}{\lim_{k \rightarrow \infty} \left(1 - \frac{1}{k}\right)^k} = \frac{1}{e^{-1}} = e.$$

(c). The radius of convergence

$$\begin{aligned} R &= \frac{1}{\lim_{k \rightarrow \infty} \frac{|a_{k+1}|}{|a_k|}} = \frac{1}{\lim_{k \rightarrow \infty} \frac{((k+1)!)^2 \cdot (2k)!}{(2k+2)! \cdot (k!)^2}} \\ &= \frac{1}{\lim_{k \rightarrow \infty} \frac{(k+1)^2}{(2k+2)(2k+1)}} = \frac{1}{\lim_{k \rightarrow \infty} \frac{(1+1/k)^2}{(2+2/k)(2+1/k)}} = 4. \end{aligned}$$

(d). The radius of convergence is 2. Let $a_n = n \left(\frac{x}{2}\right)^{n^2}$. Then

$$l = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{n} \cdot \left(\frac{|x|}{2}\right)^n = \begin{cases} 0 & |x| < 2 \\ 1 & |x| = 2 \\ +\infty & |x| > 2 \end{cases}$$

When $|x| < 2$, then the series $\sum_{n=0}^{\infty} n \left(\frac{x}{2}\right)^{n^2}$ converges absolutely. When $|x| \geq 2$, the sequence $\{a_n\}$ does not tend to 0 because $|a_n| = n \left(\frac{|x|}{2}\right)^{n^2} \geq n$ if $|x| \geq 2$. By the Divergence Test, the series $\sum_{n=0}^{\infty} n \left(\frac{x}{2}\right)^{n^2}$ diverges for $|x| \geq 2$. In conclusion, the series $\sum_{n=0}^{\infty} n \left(\frac{x}{2}\right)^{n^2}$ converges if and only if $|x| < 2$ and so the radius of convergence is 2. \square

Question 2. [20 marks] For each of the following, determine all values of x for which the given series converges.

- (a) $\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k}$.
- (b) $\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k$.
- (c) $\sum_{k=1}^{\infty} \frac{3^k x^k}{2^k (1-x)^k}, \quad x \neq 1$.
- (d) $\sum_{k=1}^{\infty} \frac{1}{kx^k}, \quad x \neq 0$.

Solution. (a). $\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{\infty} \frac{(-2)^k}{k} \left(x - \frac{1}{2}\right)^k$. The radius of convergence

$$R = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k \rightarrow \infty} \left(\frac{|-2|^k}{k}\right)^{\frac{1}{k}}} = \frac{1}{\lim_{k \rightarrow \infty} \frac{2}{\sqrt[k]{k}}} = \frac{1}{2}.$$

The ending points $x = x_0 \pm R = \frac{1}{2} \pm \frac{1}{2} = 0, 1$. When $x = 0$,

$$\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

diverges by the p -series. When $x = 1$,

$$\sum_{k=1}^{\infty} \frac{(1-2x)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converges by the alternating series test. In conclusion, the interval of convergence is $(0, 1]$.

(b). $\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k = \sum_{k=1}^{\infty} \frac{6^k}{k^3} \left(x + \frac{1}{2}\right)^k$. The radius of convergence

$$R = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k \rightarrow \infty} \left(\frac{6^k}{k^3}\right)^{\frac{1}{k}}} = \frac{1}{\lim_{k \rightarrow \infty} \frac{6}{(\sqrt[k]{k})^3}} = \frac{1}{6}.$$

The ending points $x = x_0 \pm R = -\frac{1}{2} \pm \frac{1}{6} = -\frac{2}{3}, -\frac{1}{3}$. When $x = -\frac{2}{3}$,

$$\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k = \sum_{k=1}^{\infty} \frac{(-1)^3}{k^3}$$

converges by the alternating series test. When $x = -\frac{1}{3}$,

$$\sum_{k=1}^{\infty} \frac{3^k}{k^3} (2x+1)^k = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

converges by the p -series. In conclusion, the interval of convergence is $[-\frac{2}{3}, -\frac{1}{3}]$.

(c). Let $t = \frac{3x}{2(1-x)}$. Then $\sum_{k=1}^{\infty} \frac{3^k x^k}{2^k (1-x)^k} = \sum_{k=1}^{\infty} t^k$. By the geometric series, the series $\sum_{k=1}^{\infty} \frac{3^k x^k}{2^k (1-x)^k}$ converges if and only if

$$\begin{aligned} |t| < 1 &\iff \left| \frac{3x}{2(1-x)} \right| < 1 \\ &\iff -1 < \frac{3x}{2(1-x)} < 1 \\ &\iff \begin{cases} \frac{3x}{2(1-x)} < 1 &\iff \frac{3x-2(1-x)}{2(1-x)} < 0 &\iff \frac{5x-2}{2(x-1)} > 0 &\iff x < \frac{2}{5} \text{ or } x > 1 \\ \frac{3x}{2(1-x)} > -1 &\iff \frac{3x+2(1-x)}{2(1-x)} > 0 &\iff \frac{x+2}{2(x-1)} < 0 &\iff -2 < x < 1 \end{cases} \\ &\iff -2 < x < \frac{2}{5}. \end{aligned}$$

Thus the answer is $-2 < x < \frac{2}{5}$.

(d). Let $t = \frac{1}{x}$. Then $\sum_{k=1}^{\infty} \frac{1}{kx^k} = \sum_{k=1}^{\infty} \frac{t^k}{k}$. The radius of convergence

$$R = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{\frac{1}{k}}} = 1.$$

The ending points $t = x_0 \pm R = 0 \pm 1 = \pm 1$. When $t = 1$, the series $\sum_{k=1}^{\infty} \frac{t^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by the p -series. When $t = -1$, the series $\sum_{k=1}^{\infty} \frac{t^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by the alternating series test. Thus the interval of convergence is

$$\begin{aligned} -1 \leq t < 1 &\iff -1 \leq \frac{1}{x} < 1 \\ \iff \begin{cases} \frac{1}{x} < 1 &\iff \frac{1-x}{x} < 0 &\iff \frac{x-1}{x} > 0 &\iff x < 0 \text{ or } x > 1 \\ \frac{1}{x} \geq -1 &\iff \frac{1+x}{x} \geq 0 &\iff x > 0 \text{ or } x \leq -1 \\ &\iff x \leq -1 \text{ or } x > 1. \end{cases} \end{aligned}$$

Thus the answer is $x \leq -1$ or $x > 1$. \square

Question 3. [20 marks] Using any applicable method, find the Taylor series of each of the following functions at the indicated point, and specify the interval on which the series converges to the function.

- (a) $f(x) = \cos x^2, \quad x_0 = 0.$
- (b) $f(x) = \ln\left(\frac{1+x}{1-x}\right), \quad x_0 = 0.$
- (c) $f(x) = \sqrt{x}, \quad x_0 = 1.$
- (d) $f(x) = \frac{x^2}{1-x^2}, \quad x_0 = 0.$

Solution. (a). Since $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ for $x \in \mathbb{R}$,

$$\cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (x^2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$$

and the interval is $(-\infty, \infty)$.

(b). Since

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

for $-1 < x \leq 1$, we have

$$\begin{aligned} f(x) &= \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-x)^n}{n} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n - (-1)^{2n+1} x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} x^n \\ &= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \dots = 2 \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} \end{aligned}$$

for $-1 < x < 1$.

(c). By using the binomial series,

$$f(x) = \sqrt{x} = \sqrt{1 + (x - 1)} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x - 1)^n$$

for $|x - 1| < 1$, that is, $0 < x < 2$.

(d). By using the geometric series,

$$\begin{aligned} f(x) &= \frac{x^2}{1 - x^2} = x^2 \sum_{n=0}^{\infty} (x^2)^n = x^2 \sum_{n=0}^{\infty} x^{2n} \\ &= x^2 (1 + x^2 + x^4 + x^6 + \dots) \\ &= x^2 + x^4 + x^6 + x^8 + \dots = \sum_{n=1}^{\infty} x^{2n} \end{aligned}$$

for $|x^2| < 1$, that is, $|x| < 1$. □

Question 4. [20 marks] Using the power series expansion of $\frac{1}{1 - x}$ and its derivatives, find

- (a) $\sum_{k=1}^{\infty} kx^k, \quad |x| < 1.$
- (b) $\sum_{k=1}^{\infty} \frac{k}{2^k}.$
- (c) $\sum_{k=1}^{\infty} k^2 x^k, \quad |x| < 1.$
- (d) $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^k}.$

Solution. Let $f(x) = \frac{1}{1 - x}$. Then, for $|x| < 1$,

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

Thus, for $|x| < 1$,

$$\begin{aligned} f'(x) &= 0 + 1 + 2x + 3x^2 + 4x^3 + \dots \\ \implies x f'(x) &= x + 2x^2 + 3x^3 + 4x^4 + \dots = \sum_{k=1}^{\infty} kx^k. \end{aligned}$$

It follows that

$$\sum_{k=1}^{\infty} kx^k = x f'(x) = x \left(\frac{1}{1 - x} \right)' = x \cdot (-1)(1 - x)^{-2} \cdot (-1) = \frac{x}{(1 - x)^2}.$$

This gives (a). By taking $x = \frac{1}{2}$, we have

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = 2,$$

which gives (b).

(c). and (d). Let $g(x) = \frac{x}{(1-x)^2}$. Then, from above, for $|x| < 1$,

$$\begin{aligned} g(x) &= \sum_{k=1}^{\infty} kx^k = x + 2x^2 + 3x^3 + 4x^4 + \dots \\ \implies g'(x) &= 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots \\ \implies xg'(x) &= x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \dots = \sum_{k=1}^{\infty} k^2x^k. \end{aligned}$$

Thus, for $|x| < 1$,

$$\begin{aligned} \sum_{k=1}^{\infty} k^2x^k &= xg'(x) = x \left(\frac{x}{(1-x)^2} \right)' = x \frac{1 \cdot (1-x)^2 - x \cdot 2(1-x) \cdot (-1)}{(1-x)^4} \\ &= \frac{x(1+x)}{(1-x)^3}. \end{aligned}$$

This gives (c). For (d), by taking $x = -\frac{1}{2}$, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^k} = \frac{-\frac{1}{2} \left(1 - \frac{1}{2}\right)}{\left(1 + \frac{1}{2}\right)^3} = -\frac{2}{27}.$$

□

Question 5. [20 marks]

(a) Use series to estimate the integral's value

$$\int_0^{0.1} \arctan x^2 dx$$

with an error of magnitude less than 10^{-8} .

(b) Let $f(x) = x^3 \sin x^9$. Find $f^{(30)}(0)$.

(c) Estimate $\sin \frac{1}{10}$ with an error of magnitude less than 10^{-8} .

(d) Estimate $\sqrt[3]{999}$ with an error of magnitude less than 10^{-8} .

Solution. (a). From $\arctan x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$, we have

$$\arctan x^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-2}}{2n-1}$$

$$\begin{aligned} \implies \int_0^{\frac{1}{10}} \arctan x^2 dx &= \sum_{n=1}^{\infty} \int_0^{\frac{1}{10}} \frac{(-1)^{n+1} x^{4n-2}}{2n-1} dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-1}}{(4n-1)(2n-1)} \Big|_0^{\frac{1}{10}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(4n-1)(2n-1)10^{4n-1}}. \end{aligned}$$

Let $a_n = \frac{1}{(4n-1)(2n-1)10^{4n-1}}$. Then a_n is positive, monotone decreasing and $\lim_{n \rightarrow \infty} a_n = 0$. By the alternating series estimation, from

$$a_{n+1} = \frac{1}{(4n+3)(2n+1)10^{4n+3}} < 10^{-8},$$

we have $n \geq 1$. Thus

$$\int_0^{0.1} \arctan x^2 dx \approx \frac{1}{3 \cdot 1 \cdot 10^3} = \frac{1}{3000}$$

with error less than 10^{-8} .

(b). Since $\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$,

$$\begin{aligned} f(x) &= x^3 \sin x^9 = x^3 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{9(2n-1)}}{(2n-1)!} \\ &= x^3 \left(x^9 - \frac{x^{27}}{3!} + \frac{x^{45}}{5!} - \dots \right) \\ &= x^{12} - \frac{x^{30}}{3!} + \frac{x^{48}}{5!} - \dots \end{aligned}$$

By the definition of Taylor series,

$$\frac{f^{(30)}(0)}{30!} = -\frac{1}{3!}$$

and so $f^{(30)}(0) = -\frac{30!}{3!}$.

(c). Since $\sin x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{(2n-1)!}$,

$$\sin \frac{1}{10} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)! \cdot 10^{2n-1}}.$$

Let $a_n = \frac{1}{(2n-1)! \cdot 10^{2n-1}}$. Then a_n is positive, monotone decreasing and $\lim_{n \rightarrow \infty} a_n = 0$. By the alternating series estimation, from

$$a_{n+1} = \frac{1}{(2n+1)! \cdot 10^{2n+1}} < 10^{-8},$$

we have $n \geq 3$, and so

$$\sin \frac{1}{10} \approx \frac{1}{10} - \frac{1}{3! \cdot 10^3} + \frac{1}{5! \cdot 10^5}$$

with error less than 10^{-8} .

(d). By the binomial series,

$$\begin{aligned}\sqrt[3]{999} &= (10^3 - 1)^{\frac{1}{3}} = 10 \cdot (1 - 10^{-3})^{\frac{1}{3}} = 10 \left[1 + \sum_{n=1}^{\infty} \binom{\frac{1}{3}}{n} (-10^{-3})^n \right] \\ &= 10 + \sum_{n=1}^{\infty} \binom{\frac{1}{3}}{n} \frac{(-1)^n}{10^{3n-1}}.\end{aligned}$$

Observe that, for $n \geq 1$,

$$\begin{aligned}\left| \binom{\frac{1}{3}}{n} \right| &= \left| \frac{\frac{1}{3} \cdot \left(\frac{1}{3} - 1\right) \cdots \left(\frac{1}{3} - (n-1)\right)}{n \cdot (n-1) \cdot (n-2) \cdots 1} \right| \\ &\leq \frac{\frac{1}{3} \cdot 1 \cdot 2 \cdots (n-1)}{n \cdot (n-1) \cdot (n-2) \cdots 1} = \frac{1}{3n}.\end{aligned}$$

Thus, for $n \geq 1$, the remainder

$$\begin{aligned}R_n &= \left| \sum_{k=n+1}^{\infty} \binom{\frac{1}{3}}{k} \frac{(-1)^k}{10^{3k-1}} \right| \leq \sum_{k=n+1}^{\infty} \left| \binom{\frac{1}{3}}{k} \right| \frac{1}{10^{3k-1}} \leq \sum_{k=n+1}^{\infty} \frac{1}{3k \cdot 10^{3k-1}} \\ &= \frac{1}{3(n+1) \cdot 10^{3n+2}} + \frac{1}{3(n+2) \cdot 10^{3n+5}} + \frac{1}{3(n+3) \cdot 10^{3n+8}} + \cdots \\ &\leq \frac{1}{3(n+1) \cdot 10^{3n+2}} + \frac{1}{3(n+1) \cdot 10^{3n+5}} + \frac{1}{3(n+1) \cdot 10^{3n+8}} + \cdots \\ &= \frac{1}{3(n+1) \cdot 10^{3n+2}} \left[1 + \frac{1}{10^3} + \left(\frac{1}{10^3}\right)^2 + \left(\frac{1}{10^3}\right)^3 + \cdots \right] \\ &= \frac{1}{3(n+1) \cdot 10^{3n+2}} \cdot \frac{1}{1 - \frac{1}{10^3}}.\end{aligned}$$

When $n = 2$, we have

$$R_2 \leq \frac{1}{3 \cdot 3 \cdot 10^8} \cdot \frac{1}{1 - \frac{1}{10^3}} < 10^{-8}$$

and so

$$\begin{aligned}\sqrt[3]{999} &\approx 10 + \binom{\frac{1}{3}}{1} \frac{(-1)^1}{10^{3 \cdot 1 - 1}} + \binom{\frac{1}{3}}{2} \frac{(-1)^2}{10^{3 \cdot 2 - 1}} \\ &= 10 - \frac{1}{300} + \frac{\frac{1}{3} \cdot \left(\frac{1}{3} - 1\right)}{2 \cdot 1} \cdot \frac{1}{10^5} \\ &= 10 - \frac{1}{300} - \frac{1}{9 \cdot 10^5}\end{aligned}$$

with error less than 10^{-8} . □