

Question 1(i). Observe that

$$\left| \frac{2n^2}{3n^2 + 2} - \frac{2}{3} \right| = \left| \frac{6n^2 - 2(3n^2 + 2)}{3(3n^2 + 2)} \right| = \frac{4}{9n^2 + 6} \leq \frac{4}{9n^2}.$$

Given any $\epsilon > 0$, let N be a positive integer such that $\frac{4}{9N^2} \leq \epsilon$ or $N \geq \frac{2}{3\sqrt{\epsilon}}$. Then

$$\left| \frac{2n^2}{3n^2 + 2} - \frac{2}{3} \right| \leq \frac{4}{9n^2} < \frac{4}{9N^2} \leq \epsilon$$

for $n > N$ and hence the result. \square

Question 2(ii). Observe that

$$\begin{aligned} \left| \sqrt{\frac{n}{n+1}} - 1 \right| &= \left| \frac{\sqrt{n}}{\sqrt{n+1}} - 1 \right| = \frac{|\sqrt{n} - \sqrt{n+1}|}{\sqrt{n+1}} \\ &= \frac{|(\sqrt{n} - \sqrt{n+1})(\sqrt{n} + \sqrt{n+1})|}{\sqrt{n+1}(\sqrt{n} + \sqrt{n+1})} = \frac{|n - (n+1)|}{\sqrt{n^2 + n} + n + 1} = \frac{1}{\sqrt{n^2 + n} + n + 1} \leq \frac{1}{n}. \end{aligned}$$

Given any $\epsilon > 0$, let N be a positive integer such that $\frac{1}{N} \leq \epsilon$ or $N \geq \frac{1}{\epsilon}$. Then

$$\left| \sqrt{\frac{n}{n+1}} - 1 \right| \leq \frac{1}{n} < \frac{1}{N} \leq \epsilon$$

for $n > N$ and hence the result. \square

Question 2(i). This statement is true. We prove the statement by contradiction. Suppose that $\{a_n - b_n\}$ is convergent. Let $c_n = a_n - b_n$. By Theorem 1.4.3, the sequence

$$\{a_n - c_n\} = \{a_n - (a_n - b_n)\} = \{b_n\}$$

is convergent. This contradicts to the assumption that $\{b_n\}$ is divergent and hence $\{a_n - b_n\}$ is divergent. \square

Question 2(ii). This statement is false. An counter-example is as follows. Let $a_n = \frac{1}{n}$ and let $b_n = n$. Then $\{a_n\}$ is convergent and $\{b_n\}$ is divergent. But the sequence $\{a_n b_n\} = \{1, 1, 1, \dots\}$ is convergent. \square

Question 3(i).

$$\lim_{n \rightarrow \infty} \frac{n^2 + 5n^3 - 1}{3n^3 + 6n + 4} = \lim_{n \rightarrow \infty} \frac{(n^2 + 5n^3 - 1)/n^3}{(3n^3 + 6n + 4)/n^3} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + 5 - \frac{1}{n^3}}{3 + \frac{6}{n^2} + \frac{4}{n^3}} = \frac{0 + 5 - 0}{3 + 0 + 0} = \frac{5}{3}.$$

\square

Question 3 (ii).

$$\lim_{n \rightarrow \infty} \frac{3^n + n^8}{2n^2 + 7^n} = \lim_{n \rightarrow \infty} \frac{(3^n + n^8)/7^n}{(2n^2 + 7^n)/7^n} = \lim_{n \rightarrow \infty} \frac{(\frac{3}{7})^n + \frac{n^8}{7^n}}{\frac{2n^2}{7^n} + 1} = \frac{0 + 0}{0 + 1} = 0.$$

□

Question 3 (iii).

$$\lim_{n \rightarrow \infty} \sqrt{\frac{n^4 + 4n^3 + 1}{n^3 + 2n^2}} = \lim_{n \rightarrow \infty} \sqrt{\frac{n^4}{n^3} \cdot \frac{(n^4 + 4n^3 + 1)/n^4}{(n^3 + 2n^2)/n^3}} = \lim_{n \rightarrow \infty} \sqrt{n} \cdot \sqrt{\frac{1 + \frac{4}{n} + \frac{1}{n^4}}{1 + \frac{2}{n}}} = +\infty.$$

□

Question 4 (i). Observe that

$$\frac{1}{2n} \leq \frac{1 + |\sin n|}{2n} \leq \frac{2}{2n} = \frac{1}{n}.$$

Since $\lim_{n \rightarrow \infty} \frac{1}{2n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$, we have $\lim_{n \rightarrow \infty} \frac{1 + |\sin n|}{2n} = 0$ by the Squeeze Theorem.

□

Question 4 (ii). Since

$$0 \leq \frac{2n - 5}{3n + 1} \leq \frac{2n}{3n} = \frac{2}{3}$$

for $n \geq 3$, we have

$$0 \leq \left(\frac{2n - 5}{3n + 1}\right)^n \leq \left(\frac{2}{3}\right)^n$$

for $n \geq 3$. Since $\lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = \lim_{n \rightarrow \infty} 0 = 0$, we have $\lim_{n \rightarrow \infty} \left(\frac{2n - 5}{3n + 1}\right)^n = 0$ by the Squeeze Theorem.

□

Question 5 (i). Since $\frac{e^n}{n^{100}} > 0$ and $\lim_{n \rightarrow \infty} \frac{n^{100}}{e^n} = 0$, we have $\lim_{n \rightarrow \infty} \frac{e^n}{n^{100}} = +\infty$ by the Reciprocal Rule

□

Question 5 (ii). Since $\frac{n}{\ln \frac{1}{n+2}} = -\frac{n}{\ln(n+2)} < 0$ and

$$\lim_{n \rightarrow \infty} -\frac{\ln(n+2)}{n} = \lim_{n \rightarrow \infty} -\frac{\frac{1}{n+2}}{1} = 0,$$

we have $\lim_{n \rightarrow \infty} \frac{n}{\ln \frac{1}{n+2}} = -\infty$ by the Reciprocal Rule.

□

Question 6 (a).

$$\lim_{n \rightarrow \infty} \sqrt[3]{\frac{2n^4 + n + 1}{16n^4 + n^2 + 2}} = \lim_{n \rightarrow \infty} \sqrt[3]{\frac{(2n^4 + n + 1)/n^4}{(16n^4 + n^2 + 2)/n^4}}$$

$$= \lim_{n \rightarrow \infty} \sqrt[3]{\frac{2 + \frac{1}{n^3} + \frac{1}{n^4}}{16 + \frac{1}{n^2} + \frac{2}{n^4}}} = \sqrt[3]{\frac{2 + 0 + 0}{16 + 0 + 0}} = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}.$$

□

Question 6 (b).

$$\lim_{n \rightarrow \infty} \left(3 + \ln \left(\cos \frac{1}{\sqrt{n}} \right) + \frac{n^2}{1.1^n} \right) = 3 + \ln(\cos 0) + 0 = 3.$$

□

Question 6 (c).

$$\lim_{n \rightarrow \infty} \frac{n^4 + 8^n}{9^n + n + 8^n} = \lim_{n \rightarrow \infty} \frac{(n^4 + 8^n)/9^n}{(9^n + n + 8^n)/9^n} = \lim_{n \rightarrow \infty} \frac{\frac{n^4}{9^n} + \left(\frac{8}{9}\right)^n}{1 + \frac{n}{9^n} + \left(\frac{8}{9}\right)^n} = \frac{0 + 0}{1 + 0 + 0} = 0.$$

□

Question 6 (d).

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{3}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{3^{\frac{1}{n}}} = \frac{1}{3^0} = 1.$$

□

Question 6 (e).

$$\begin{aligned} \lim_{n \rightarrow \infty} (\sqrt{3n-2} - \sqrt{3n-3}) &= \lim_{n \rightarrow \infty} \frac{(\sqrt{3n-2} - \sqrt{3n-3})(\sqrt{3n-2} + \sqrt{3n-3})}{\sqrt{3n-2} + \sqrt{3n-3}} \\ &= \lim_{n \rightarrow \infty} \frac{(3n-2) - (3n-3)}{\sqrt{3n-2} + \sqrt{3n-3}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{3n-2} + \sqrt{3n-3}} = 0. \end{aligned}$$

□

Question 6 (f). Since $0 \leq \frac{3 + (-1)^n}{5} \leq \frac{4}{5}$, we have $0 \leq \left(\frac{3 + (-1)^n}{5}\right)^n \leq \left(\frac{4}{5}\right)^n$.

Since $\lim_{n \rightarrow \infty} \left(\frac{4}{5}\right)^n = \lim_{n \rightarrow \infty} 0 = 0$, we have $\lim_{n \rightarrow \infty} \left(\frac{3 + (-1)^n}{5}\right)^n = 0$ by the Squeeze Theorem.

□

Question 6 (g).

$$\lim_{n \rightarrow \infty} \frac{7^n + \ln n - n!}{n! + n^2} = \lim_{n \rightarrow \infty} \frac{(7^n + \ln n - n!)/n!}{(n! + n^2)/n!} = \lim_{n \rightarrow \infty} \frac{\frac{7^n}{n!} + \frac{\ln n}{n!} - 1}{1 + \frac{n^2}{n!}} = \frac{0 + 0 - 1}{1 + 0} = -1.$$

□

Question 6 (h).

$$\lim_{n \rightarrow \infty} \frac{n^{100} 100^n}{n!} = \lim_{n \rightarrow \infty} \frac{n^{100}}{2^n} \cdot \frac{2^n \cdot 100^n}{n!} = \lim_{n \rightarrow \infty} \frac{n^{100}}{2^n} \cdot \frac{200^n}{n!} = 0 \cdot 0 = 0.$$

□

Question 6 (i). Observe that

$$\lim_{n \rightarrow \infty} \frac{\ln n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{2}n^{-\frac{1}{2}}} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt{n}} = 0.$$

Since

$$-\frac{\ln n}{\sqrt{n}} \leq \frac{(-1)^{n+1} \ln n}{\sqrt{n}} \leq \frac{\ln n}{\sqrt{n}},$$

we have $\lim_{n \rightarrow \infty} \frac{(-1)^{n+1} \ln n}{\sqrt{n}} = 0$ by the Squeeze Theorem. \square

Question 6 (j).

$$\lim_{n \rightarrow \infty} (3^n + 4^n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} (4^n)^{\frac{1}{n}} \cdot \left(\left(\frac{3}{4} \right)^n + 1 \right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} 4 \cdot \left(\left(\frac{3}{4} \right)^n + 1 \right)^{\frac{1}{n}} = 4 \cdot (0+1)^0 = 4.$$

\square

Question 6 (k).

$$\lim_{n \rightarrow \infty} n \sin \frac{3}{n} = \lim_{n \rightarrow \infty} 3 \cdot \frac{\sin \frac{3}{n}}{\frac{3}{n}} = \lim_{x \rightarrow 0} 3 \cdot \frac{\sin x}{x} = 3 \cdot 1 = 3,$$

where $x = \frac{3}{n}$. \square

Question 6 (l).

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n} \right)^{2n} = \lim_{n \rightarrow \infty} \left(\left(1 + \frac{-\frac{1}{3}}{n} \right)^n \right)^2 = \left(e^{-\frac{1}{3}} \right)^2 = e^{-\frac{2}{3}}.$$

\square

Question 6 (m).

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^3} \right)^{n^3+2} &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^3} \right)^{n^3} \cdot \left(1 + \frac{1}{n^3} \right)^2 \\ &= \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m} \right)^m \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^3} \right)^2 = e^1 \cdot (1+0)^2 = e, \end{aligned}$$

where $m = n^3$. \square

Question 7. Since S and T are bounded sets, there exist numbers m_1 , m_2 , M_1 and M_2 such that $m_1 \leq x \leq M_1$ for all $x \in S$ and $m_2 \leq y \leq M_2$ for all $y \in T$. Let z be any number in $S \cup T$. Then $z \in S$ or T . If $z \in S$, then $m_1 \leq z \leq M_1$. Otherwise, $z \in T$ and $m_2 \leq z \leq M_2$. In both cases, we have $\min\{m_1, m_2\} \leq z \leq \max\{M_1, M_2\}$ and so $S \cup T$ is bounded, where $\max\{M_1, M_2\}$ is an upper bound and $\min\{m_1, m_2\}$ is a lower bound. \square

Question 8 (i). \implies Suppose that $\{a_n\}$ is bounded. There exist numbers m and M such that $m \leq a_n \leq M$ for all n . Thus $0 \leq |a_n| \leq \max\{|m|, |M|\}$ for all n and so $\{|a_n|\}$ is a bounded sequence, where $\max\{|m|, |M|\}$ is an upper bound and 0 is a lower bound.

\Leftarrow Suppose that $\{|a_n|\}$ is bounded. Then $\{|a_n|\}$ has an upper bound, that is there exists a number M such that $|a_n| \leq M$ for all n . It follows that $-M \leq a_n \leq M$ for all n . Thus $\{a_n\}$ is a bounded sequence, where M is an upper bound and $-M$ is a lower bound. \square

Question 8 (ii). Since $\{b_n\}$ is bounded, the sequence $\{|b_n|\}$ is bounded by (i) and so there exists a positive number M such that $|b_n| \leq M$ for all n . Given any $\epsilon > 0$, since $\lim_{n \rightarrow \infty} a_n = 0$, there exists a positive integer N such that $|a_n| = |a_n - 0| < \frac{\epsilon}{M}$ for all $n > N$. It follows that

$$|a_n b_n - 0| = |a_n b_n| = |a_n| \cdot |b_n| \leq |a_n| \cdot M < \frac{\epsilon}{M} \cdot M = \epsilon$$

for all $n > N$ and so $\lim_{n \rightarrow \infty} a_n b_n = 0$. \square