Solutions to Tutorial 11

Question 1 (i). $f'(x) = 2e^{2x}$, $f''(x) = 2^2 e^{2x}$ and in general $f^{(k)}(x) = 2^k e^{2x}$. Thus

$$e^{2x} \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(3)}{k!}(x-3)^k = \sum_{k=0}^{\infty} \frac{2^k e^6}{k!}(x-3)^k = e^6 \sum_{k=0}^{\infty} \frac{2^k}{k!}(x-3)^k.$$

□

Question 1 (ii). Since $f(x) = \cos x$, we have

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = f(x) = \cos x.$$

In general,

$$f^{(k)}(x) = \begin{cases} \cos x & \text{if} \quad k = 4l \\ -\sin x & \text{if} \quad k = 4l+1 \\ -\cos x & \text{if} \quad k = 4l+2 \\ \sin x & \text{if} \quad k = 4l+3 \end{cases}$$

and so

$$f^{(k)}\left(\frac{\pi}{3}\right) = \begin{cases} \frac{1}{2} & \text{if} \quad k = 4l \\ -\frac{\sqrt{3}}{2} & \text{if} \quad k = 4l+1 \\ -\frac{1}{2} & \text{if} \quad k = 4l+2 \\ \frac{\sqrt{3}}{2} & \text{if} \quad k = 4l+3 \end{cases}$$

Thus

$$\cos x \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(\frac{\pi}{3})}{k!}\left(x - \frac{\pi}{3}\right)^k,$$

where $f^{(k)}(\frac{\pi}{3})$ is given above. □

Question 2. From

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n},$$

$$\ln(1+2x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(2x^2\right)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2^n}{n} x^{2n}$$

for $|x| < \frac{1}{2}$. □

Question 3. Let $f(x) = \cos x$. Then

$$f^{(k)}(x) = \begin{cases} \cos x & \text{if} \quad k = 4l \\ -\sin x & \text{if} \quad k = 4l+1 \\ -\cos x & \text{if} \quad k = 4l+2 \\ \sin x & \text{if} \quad k = 4l+3 \end{cases}$$

By the Taylor Formula, we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}.$$

Observe that

$$0 \leq |R_n(x)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} |x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}.$$

By the Standard Limits, $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$. Thus $\lim_{n\to\infty} |R_n(x)| = 0$ by the Squeeze Theorem and so $\lim_{n\to\infty} R_n(x) = 0$. Hence

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \lim_{n\to\infty} f(x) - R_n(x) = f(x) - 0 = f(x).$$

It follows that

$$\cos x = f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

□

Question 4.

$$\int_0^{0.2} \sin x^2 dx = \int_0^{\frac{1}{5}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x^2)^{2n-1}}{(2n-1)!} dx = \sum_{n=1}^{\infty} \int_0^{\frac{1}{5}} (-1)^{n+1} \frac{x^{4n-2}}{(2n-1)!} dx$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(\frac{1}{5}\right)^{4n-1}}{(2n-1)! \cdot 4n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)! \cdot (4n-1) \cdot 5^{4n-1}},$$

where $\int_0^{\frac{1}{5}} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \int_0^{\frac{1}{5}}$ because the series of functions $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n-2}}{(2n-1)!}$ converges uniformly on $\left[0, \frac{1}{5}\right]$ by Theorem 7.5. Let $a_n = \frac{1}{(2n-1)! \cdot (4n-1) \cdot 5^{4n-1}}$. Then a_n is positive, monotone decreasing and $\lim_{n\to\infty} a_n = 0$. By applying the alternating series estimation, from

$$a_{n+1} = \frac{1}{(2n+1)! \cdot (4n+3) \cdot 5^{4n+3}} < 10^{-8},$$

we have $n \geq 2$. Thus

$$\int_0^{0.2} \sin x^2 dx \approx \frac{1}{3 \cdot 5^3} - \frac{1}{3! \cdot 7 \cdot 5^7} = \frac{1}{375} - \frac{1}{3281250} \approx 0.002666362$$

with error $< 10^{-8}$.

□

Question 5 (i). Recall that

$$\arctan y = y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots,$$
$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots,$$
$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots.$$

$$\lim_{y\to 0} \frac{\arctan y - \sin y}{y^3 \cos y} = \lim_{y\to 0} \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)}{y^3\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right)}$$
$$= \lim_{y\to 0} \frac{\left(-\frac{1}{3} + \frac{1}{3!}\right) + \left(\frac{1}{5} - \frac{1}{5!}\right) y^2 + \cdots}{1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots} = -\frac{1}{3} + \frac{1}{3!} = -\frac{1}{6}.$$

□

Question 5 (ii). Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots.$$

$$\lim_{x\to\infty} x^2(e^{-1/x^2} - 1) = \lim_{x\to\infty} x^2\left(1 - \frac{1}{x^2} + \frac{1}{2!x^4} - \cdots - 1\right) = \lim_{x\to\infty} -1 + \frac{1}{2!x^2} - \cdots = -1.$$

□