2002/2003 Semester I MA2108 Advanced Calculus II
Solutions to Tutorial 6

*Question 1.* Let $a_n = (-1)^n \frac{\cos n}{2^n}$. Then $|a_n| \leq \frac{1}{2^n}$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent by the geometric series, the series $\sum_{n=1}^{\infty} \left|(-1)^n \frac{\cos n}{2^n}\right|$ is convergent by the comparison test and so the series $\sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{2^n}$ is absolutely convergent. $\square$

*Question 2 (a).* This series is conditionally convergent because it is convergent by the alternating series test and the series $\sum_{n=1}^{\infty} \left|(-1)^n \frac{3}{2n+1}\right|$ is divergent by the limit comparison test with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. $\square$

*Question 2 (b).* Let $a_n = (-1)^n \frac{n}{4n+3}$. Then $\lim_{n\to\infty} a_{2n-1} = -\frac{1}{4}$ and $\lim_{n\to\infty} a_{2n} = \frac{1}{4}$. Thus the limit of $(-1)^n \frac{n}{4n+3}$ does not exist and so the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{4n+3}$ is divergent by the divergence test. $\square$

*Question 2 (c).* Let $a_n = (-1)^n \left(\frac{1+2n}{3+4n}\right)^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1+2n}{3+4n} = \frac{2}{4} = \frac{1}{2} < 1.$$

Thus the positive series $\sum_{n=1}^{\infty} |a_n|$ is convergent by the simplified root test and so the series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1+2n}{3+4n}\right)^n$ is absolutely convergent. $\square$

*Question 2 (d).* Observe

$$\left|(-1)^{n+1} \frac{\cos n}{n(\ln n)^2}\right| \leq \frac{1}{n(\ln n)^2}.$$

Let $f(x) = \frac{1}{x(\ln x)^2}$. Then $f(x)$ is positive and monotone decreasing on $[2, +\infty)$. Since the integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx \underset{dy=\frac{1}{x}dx}{\overset{y=\ln x}{=\!=\!=}} \int_{\ln 2}^{\infty} \frac{1}{y^2} dy = -\frac{1}{y}\bigg|_{\ln 2}^{\infty} = \frac{1}{\ln 2}$$

is convergent, the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ is convergent by the integral test. By the comparison test, the series

$$\sum_{n=2}^{\infty}\left|(-1)^{n+1}\frac{\cos n}{n(\ln n)^2}\right|$$

is convergent and so the series

$$\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\cos n}{n(\ln n)^2}$$

is absolutely convergent. $\square$

*Question 3.* We use alternating series test estimation for solving this question. Let $a_n = \dfrac{1}{n^5}$. From $a_{n+1} = \dfrac{1}{(n+1)^5} < 0.001$, we have $n+1 > \sqrt[5]{1000}$ or $n \geq 3$ and so

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^5} \approx 1 - \frac{1}{2^5} + \frac{1}{3^5} \approx 0.9729$$

with error less than 0.001. $\square$

*Question 4 (a).* $F(x) = \displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{nx} = (e^x)^x = e^{x^2}$ $\square$

*Question 4 (b).* It does not converge pointwise because $\displaystyle\lim_{n\to\infty}(-1)^{n+1}$ does not exist. $\square$

*Question 4 (c).*

$$F(x) = \lim_{n\to\infty}\frac{x^{2n}}{1+x^{2n}} = \begin{cases} \frac{0}{1+0} = 0 & \text{if} \quad 0 \leq x < 1 \\ \frac{1}{2} & \text{if} \quad\quad x = 1 \end{cases}$$

$\square$

*Question 5.* For $\epsilon = 1$, since $F_n(x)$ converges uniformly to $F(x)$ on $I$, there exists $N$ such that

$$|F_n(x) - F(x)| < 1$$

for all $x \in I$ and $n > N$. Thus $|F_{N+1}(x) - F(x)| < 1$ or

$$F_{N+1}(x) - 1 < F(x) < F_{N+1}(x) + 1$$

for all $x \in I$ and so

$$|F(x)| < \max\{|F_{N+1}(x)+1|, |F_{N+1}(x)-1|\} \leq |F_{N+1}(x)| + 1 \leq M_{N+1} + 1$$

for all $x \in I$. Let $M = M_{N+1} + 1$. Then $|F(x)| \leq M$ for all $x \in I$.

$\square$