Solutions to Tutorial 8

Question 1 (i). Let $F_n(x) = \dfrac{n+e^x}{n+x^2}$. Then the limiting function

$$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{n+e^x}{n+x^2} = \lim_{n\to\infty} \frac{1+\frac{e^x}{n}}{1+\frac{x^2}{n}} = 1$$

and

$$0 \le T_n = \sup_{0\le x\le 1} |F_n(x) - F(x)| = \sup_{0\le x\le 1} \left| \frac{n+e^x}{n+x^2} - 1 \right|$$

$$= \sup_{0\le x\le 1} \frac{|e^x - x^2|}{n^2+x^2} \le \sup_{0\le x\le 1} \frac{e^x+x^2}{n^2+x^2} \le \frac{e+1}{n^2}$$

Since $\lim_{n\to\infty} \dfrac{e+1}{n^2} = \lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze Theorem. Thus $\{F_n\}$ converges uniformly to $F(x)$ and so

$$\lim_{n\to\infty} \int_0^1 \frac{n+e^x}{n+x^2} dx = \int_0^1 \lim_{n\to\infty} F_n(x) dx = \int_0^1 1 dx = 1.$$

□

Question 1 (ii). Let $F_n(x) = \left(\dfrac{x^2+1}{8}\right)^n \sin nx$. Then

$$0 \le |F_n(x)| = \left| \left(\frac{x^2+1}{8}\right)^n \sin nx \right| \le \left(\frac{2^2+1}{8}\right)^n = \left(\frac{5}{8}\right)^n$$

for $1 \le x \le 2$. Since $\lim_{n\to\infty} \left(\dfrac{5}{8}\right)^n = \lim_{n\to\infty} 0 = 0$, the limiting function $F(x) = 0$ for $1 \le x \le 2$ and

$$0 \le T_n = \sup_{1\le x\le 2} |F_n(x) - F(x)| = \sup_{1\le x\le 2} |F_n(x)| \le \left(\frac{5}{8}\right)^n$$

Since $\lim_{n\to\infty} \left(\dfrac{5}{8}\right)^n = 0$, $\lim_{n\to\infty} T_n = 0$ by the Squeeze Theorem and so $\{F_n\}$ converges uniformly to $F(x)$. Thus

$$\lim_{n\to\infty} \int_1^2 \left(\frac{x^2+1}{8}\right)^n \sin nx dx = \int_1^2 \lim_{n\to\infty} F_n(x) dx = \int_1^2 0 dx = 0.$$

□

Question 2. Let $f_k(x) = \dfrac{(-1)^k x^k}{1+x^{2k}}$ for $0 < x < \dfrac{2}{3}$. Since

$$|f_k(x)| = \left|\frac{(-1)^k x^k}{1+x^{2k}}\right| \le \frac{\left(\frac{2}{3}\right)^k}{1} = \left(\frac{2}{3}\right)^k$$

for $0 < x < \dfrac{2}{3}$ and the series $\displaystyle\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$ converges by the geometric series, the series of functions $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{1+x^{2k}}$ converges uniformly on $\left(0, \dfrac{2}{3}\right)$ by the Weierstrass M-test. Thus the function $F(x) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^k x^k}{1+x^{2k}}$ is continuous on the interval $\left(0, \dfrac{2}{3}\right)$. □

Question 3. Let $f_n(x) = \dfrac{x^n(1-x^2)}{\sqrt{1+x}}$ for $0 \le x \le \dfrac{1}{2}$. Since

$$|f_n(x)| = \frac{|x|^n|1-x^2|}{\sqrt{1+x}} \le \frac{\left(\frac{1}{2}\right)^n \cdot 1}{1} = \left(\frac{1}{2}\right)^n$$

for $0 \le x \le \dfrac{1}{2}$ and the series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$ is convergent by the geometric series, the series of functions $\displaystyle\sum_{n=0}^{\infty}\frac{x^n(1-x^2)}{\sqrt{1+x}}$ converges uniformly on $[0, \frac{1}{2}]$ by the M-test. Note that each $f_n(x)$ is Riemann integrable on $[0, \frac{1}{2}]$. We have

$$\sum_{n=0}^{\infty}\int_0^{\frac{1}{2}}\frac{x^n(1-x^2)}{\sqrt{1+x}}dx = \int_0^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{x^n(1-x^2)}{\sqrt{1+x}}dx = \int_0^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}x^n\right)\cdot\frac{1-x^2}{\sqrt{1+x}}dx$$

$$= \int_0^{\frac{1}{2}}\frac{1}{1-x}\cdot\frac{1-x^2}{\sqrt{1+x}}dx = \int_0^{\frac{1}{2}}\sqrt{1+x}dx = \frac{2}{3}\left(1+x\right)^{\frac{3}{2}}\Big|_0^{\frac{1}{2}} = \frac{2}{3}\left[\left(\frac{3}{2}\right)^{\frac{3}{2}} - 1\right] = \sqrt{\frac{3}{2}} - \frac{2}{3}.$$

□

Question 4 (i). Since

$$|a_k \sin kx| \le |a_k|$$

for $x \in (-\infty, +\infty)$ and $\displaystyle\sum_{k=1}^{\infty}|a_k|$ is convergent, the series of functions $\displaystyle\sum_{k=1}^{\infty}a_k\sin kx$ converges uniformly on $(-\infty, +\infty)$ by the M-test. □

Question 4 (ii). By (i), the series of functions $\sum_{k=1}^{\infty} a_k \sin kx$ converges uniformly on $[0, 2\pi]$. Since each $a_k \sin kx$ is Riemann integrable, we have

$$\int_0^{2\pi} \sum_{k=1}^{\infty} a_k \sin kx dx = \sum_{k=1}^{\infty} \int_0^{2\pi} a_k \sin kx dx = \sum_{k=1}^{\infty} 0 = 0.$$

□

Question 5. From $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for any x, we have

$$e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}.$$

Since

$$\left| \frac{(-1)^n x^{3n}}{n!} \right| \leq \frac{1}{n!}$$

for $0 \leq x \leq 1$ and the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent by the ratio test, the series of functions $\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$ converges uniformly to e^{-x^3} on $[0, 1]$ by the M-test. Note that each $\frac{(-1)^n x^{3n}}{n!}$ is Riemann integrable. Thus

$$\int_0^1 e^{-x^3} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{3n}}{n!} dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!(3n+1)}.$$

Let $a_n = \frac{1}{n!(3n+1)}$. Then the sequence $\{a_n\}$ is positive, monotone decreasing and $\lim_{n\to\infty} a_n = 0$. By applying the alternating test estimation, from $a_{n+1} < 0.001$ or $(n+1)!(3n+4) \geq 1000$, we have $n \geq 4$ and so

$$\int_0^1 e^{-x^3} dx \approx 1 - \frac{1}{1! \cdot 4} + \frac{1}{2! \cdot 7} - \frac{1}{3! \cdot 10} + \frac{1}{4! \cdot 13} = 1 - \frac{1}{4} + \frac{1}{14} - \frac{1}{60} + \frac{1}{312}$$

with error less than 0.001. □