NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2002-2003

# MA2108 ADVANCED CALCULUS II

November 2002 — Time allowed : 2 hours

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**INSTRUCTIONS TO CANDIDATES**

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.
3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

## SECTION A

*Answer* **all** *the questions in this section. Section A carries a total of 60 marks.*

**Question 1** [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\sqrt{n^2+2n}-n\right\}$.

(b) $\left\{(6^n+8^n)^{\frac{1}{n}}\right\}$.

(c) $\left\{\left(\dfrac{3n}{3n-2}\right)^{2n+\sqrt{n}}\right\}$.

(d) $\left\{\dfrac{n^{100}\cdot 100^n\cdot\cos n}{n!}\right\}$.

**Question 2** [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+3n-1}$.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(1+2\ln n)}$.

(c) $\displaystyle\sum_{n=1}^{\infty}5^n\left(1-\frac{2}{n+3}\right)^{n^2}$.

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{n}$.

**Question 3** [10 marks]

Find the radius of convergence of each of the following power series. Justify your answer.

(a) $$\sum_{k=1}^{\infty} \left(1 - \frac{2}{k}\right)^{k^2} (x-1)^k.$$

(b) $$\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}(2x+1)^k.$$

**Question 4** [18 marks]

(a) Find limit inferior and limit superior of each of the following sequences.

(i) $$\left\{\left[(-1)^n - \frac{1}{2}\right]^n\right\}.$$

(ii) $$\left\{\left(1 + \frac{1}{n}\right)\left(1 + \sin\frac{n\pi}{4}\right)^{\frac{1}{n}}\right\}.$$

(b) Is the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{2\ln n + 1}{3\sqrt{n}}$ absolutely convergent, conditionally convergent or divergent? Justify your answer.

## SECTION B

*Answer not more than* **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

**Question 5** [20 marks]

(a) Evaluate $\lim_{n\to\infty} \int_0^{\frac{1}{2}} \frac{x^n \sin nx}{1+x^n}\, dx$. Justify your answer.

(b) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(1-2x)^n}{3n+1}$.

Justify your answer.

(c) Let $\{a_n\}$ be a bounded sequence of real numbers. Show that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \leq 1.$$

**Question 6** [20 marks]

(a) Consider the function

$$f(x) = \sum_{n=1}^{\infty} x^n e^{-nx}.$$

Is $f(x)$ continuous on $[0, +\infty)$? Justify your answer.

(b) Consider the sequence $\{x_n\}$ defined recursively by

$$x_1 = 2, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right), \quad \text{for} \quad \text{n} \geq 1.$$

Show that $\{x_n\}$ converges, and find its limit.

(c) Show that the series $\sum_{k=1}^{\infty} \frac{1}{k^{1+2x}}$ does not converge uniformly on $(0, +\infty)$.

**Question 7** [20 marks]

(a) Let $f(x) = x^5 \sin(x^9)$. Find $f^{(48)}(0)$.

(b) Let $A$ and $B$ be two non-empty bounded set of real numbers. Define $A + B = \{a + b \mid a \in A,\ b \in B\}$. Prove that

$$\inf A + \inf B = \inf(A + B).$$

(c) Show that the series of functions $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n+1)^x}$ converges uniformly on $[1, +\infty)$.